Introduction to Elliptic Curves

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Finding integer/rational solutions to polynomial equations

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Diophantus, Pythagoras

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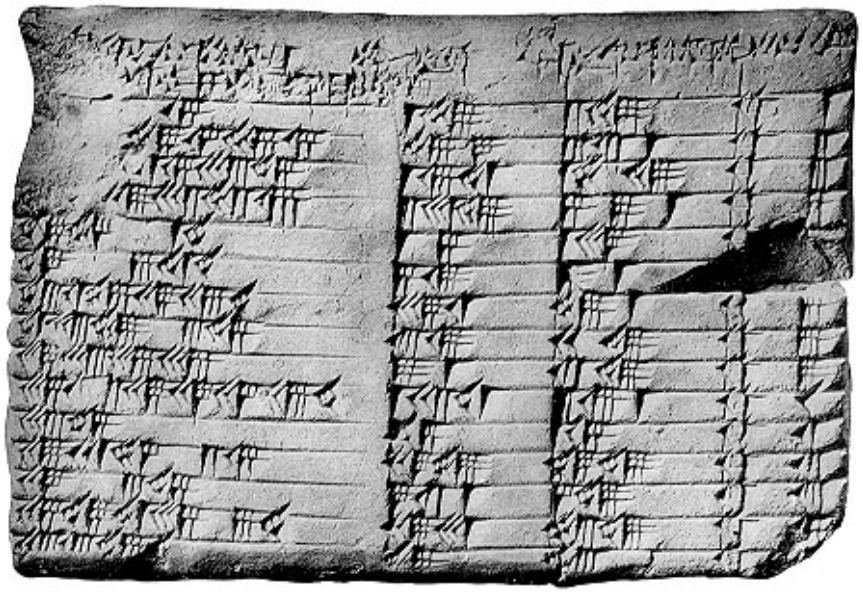
Babylonian tablet Plimpton 322



Mathematical Cuneiform Tablets

Otto Neugebauer & Abraham Sachs, 1945

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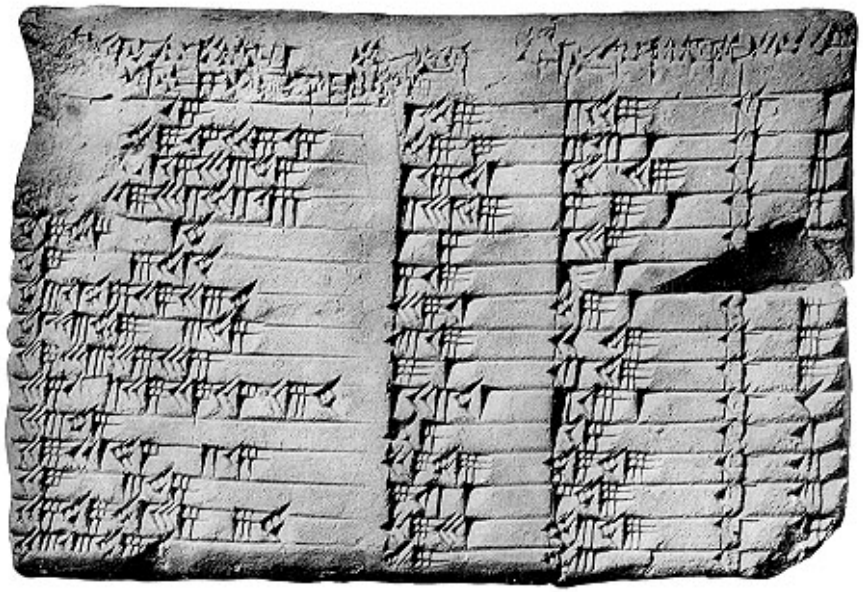

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N.A.B.U. (Nouvelles Assyriologiques Brèves et Utilitaires) Sept 2017

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$\rightsquigarrow$ algorithm that tests a finite number of candidate rational roots

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Example

Find all rational solutions of $x^2 + y^2 = 1$

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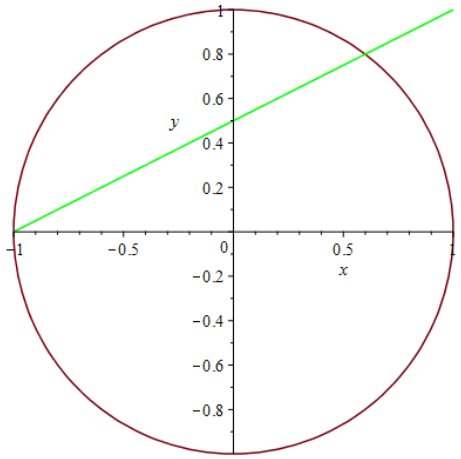

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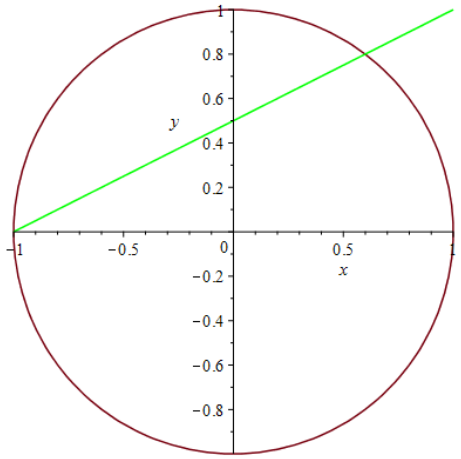

Key observation I If $P = (x, y)$ is another rational point on the unit circle, then the slope s of the line joining $(-1, 0)$ and P is rational.

Key observation II: The converse is also true!

If you take a line with rational slope through $(-1, 0)$ that intersects the unit circle, then the point of intersection is a rational point.

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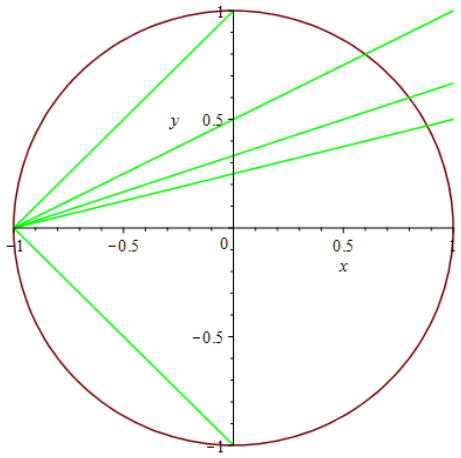

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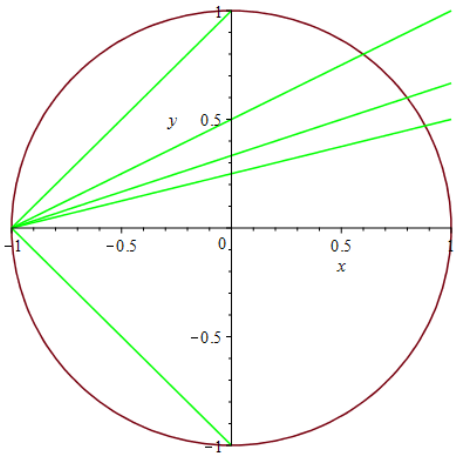

Proof Suppose we take a line of rational slope s through $(-1, 0)$,

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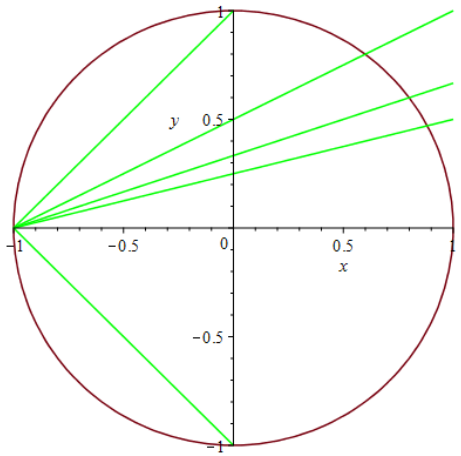

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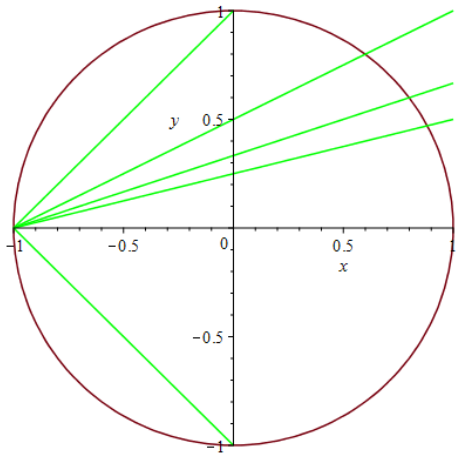

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Substitution of (1) in (2) yields the quadratic equation

$$x^2 + s^2(x + 1)^2 = 1 \tag{3}$$

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The correspondence between slopes s and points (x,y) is:

$$s \mapsto \left(-\frac{s^2 - 1}{s^2 + 1}, \frac{2s}{s^2 + 1} \right) \tag{4}$$

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5. (think modulo powers of 3)

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If BSD was true, then we would have a method to find all rational solutions to a cubic equation in two variables and determine whether it has finitely many of infinitely many rational solutions.

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5. It is now easy to describe an amazing structure, exhibited by the set of rational points on an elliptic curve

The graph of an elliptic curve

Typical graphs of elliptic curves in $\mathbb{R}^2$:

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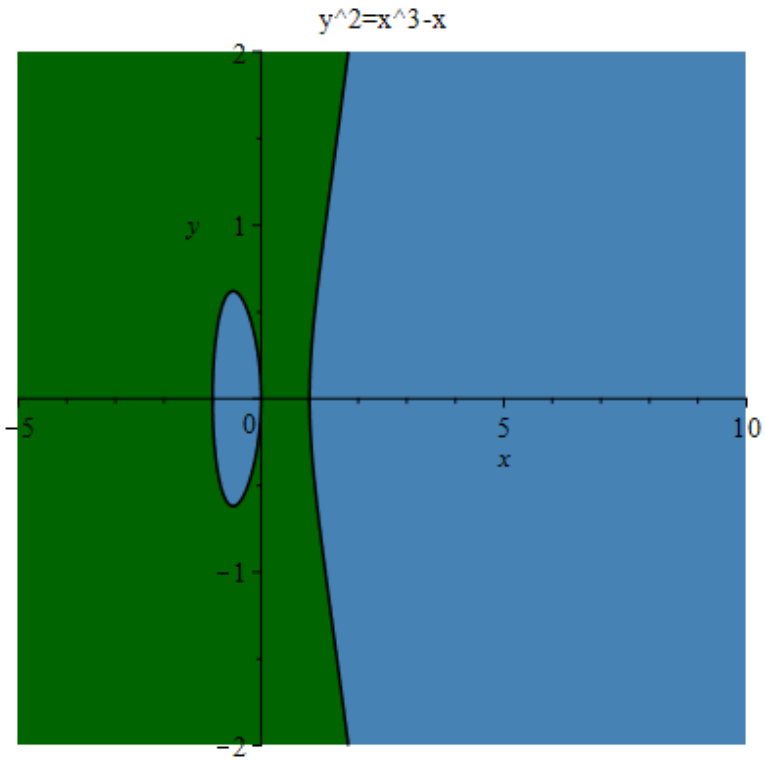

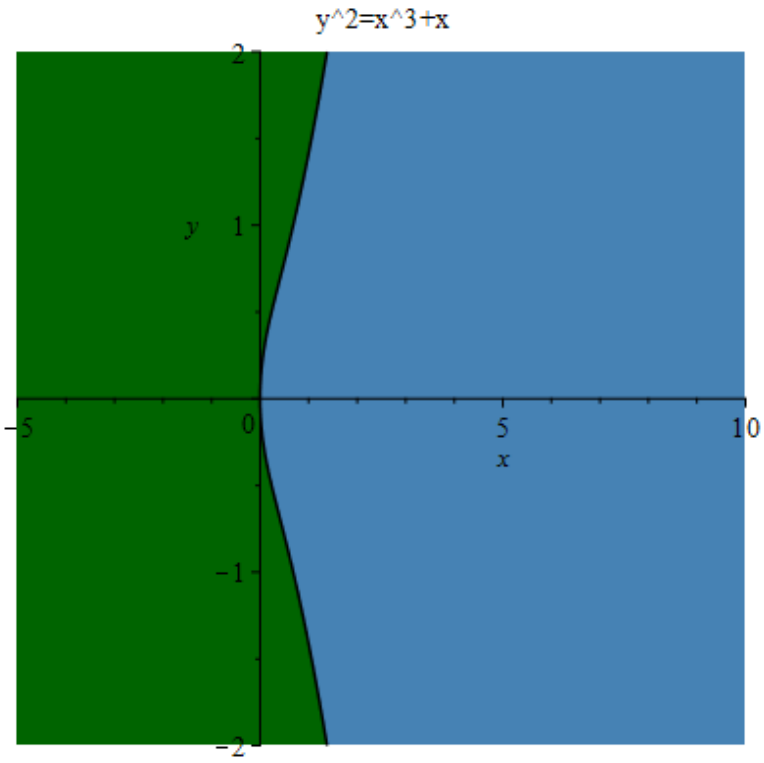

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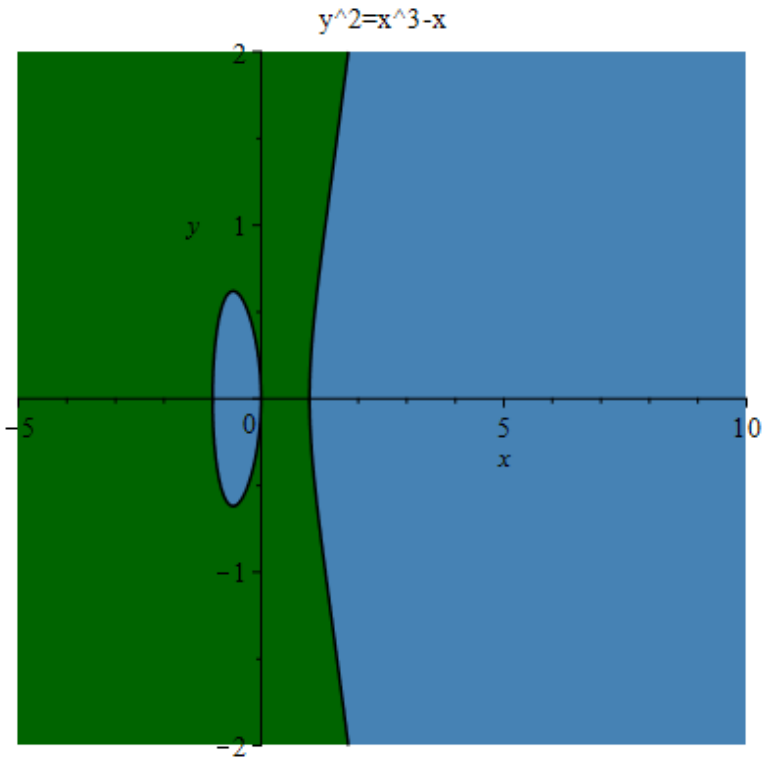

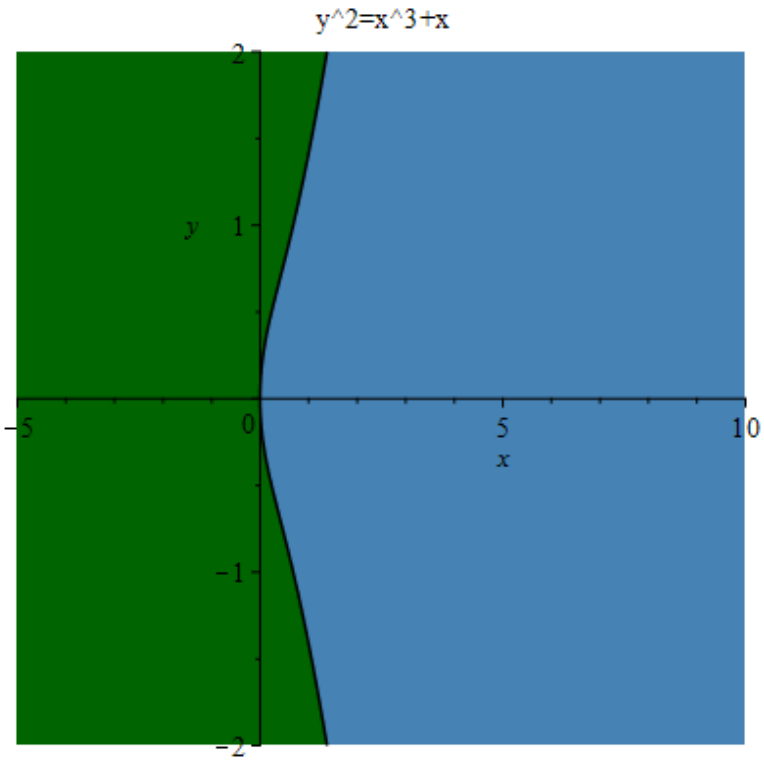

(depending on whether the x-part has 1 real root or 3 real roots)

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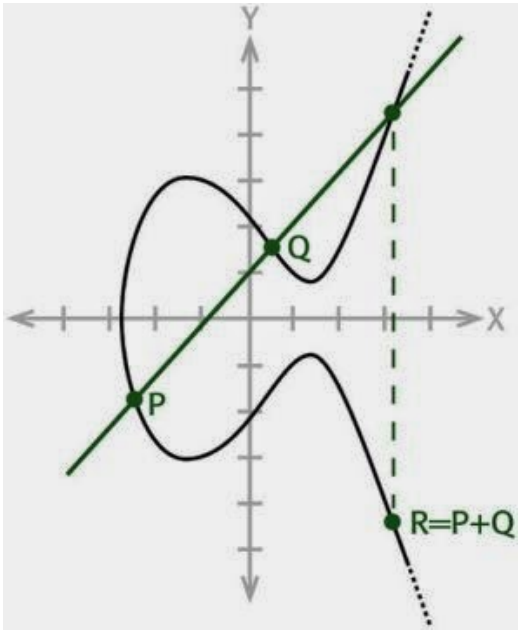

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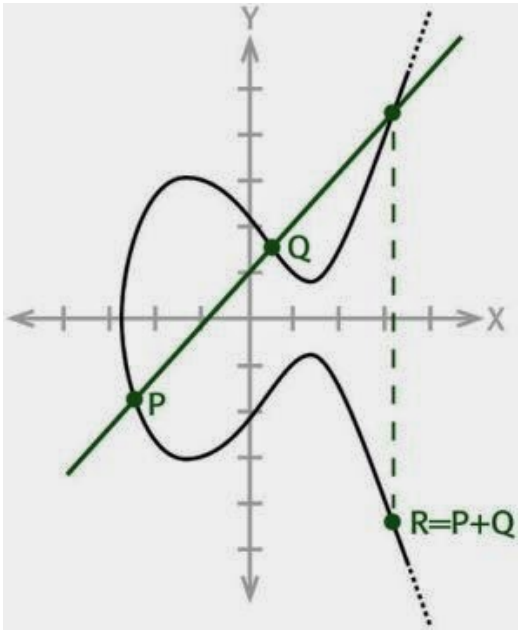

Fact

This law of addition endows the set of r.p. on E, together with the point at ∞, with the structure of an abelian group.

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- The rank is a fundamental arithmetic invariant of an elliptic curve

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- In Number Theory, we are interested in the behavior of the rank

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6. The BSD conjecture would give such an algorithm.

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6. Thus N_p/p should be one most of the time
7. However, if E has lot of rational point on it, then reducing these mod p would give a lot of points mod p on the elliptic curve.
8. BSD hypothesized that if the rank of an elliptic curve E is large, then on the average (as p varies) one should notice E having more than p points modulo p.

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This is called the strong form of the BSD conjecture
Remark If you prove BSD (weak/strong), you win a million $!

BSD = fundamental problem in NT

BSD = Birch and Swinnerton-Dyer conjecture

Named after Bryan Birch and Peter Swinnerton-Dyer, who developed it in the 1960's with the help of machine computation

- Describe BSD in elementary terms
- Explain why BSD is so important
- Mention what is known about it

The 7 Clay Millennium Prize Problems

Posed in 2000, in celebration of the new millennium

$ 1 M prize (each)

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BSD was discovered in its elementary formulation.

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Questions on Bachet's formula

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Main Idea

If we start with a point (a, b) on the elliptic curve (6), then the formula gives, as the next point, the point of intersection of the curve with the tangent line to the curve (6)

- Do you think this sequence of points can be periodic?
- Is it always the case that the tangent intersects the curve at exactly one point?
- What is going on with the bad case $y = 0$?

Elliptic Curves over $\mathbb{F}_p$

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$E(\mathbb{F}_7)$ is an abelian group of order 9, isomorphic $\mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_9$

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$\Delta = 4 \cdot 5^3 + 27 \cdot 2^2 \equiv 4 \cdot (-2)^3 + 6 \cdot 2^2 \equiv -8 \equiv 6 \neq 0 (mod 7)$

$E(\mathbb{F}_7) = \{\mathbf{0}, \underbrace{[0,3], [0,4]}_{[0,\pm 3]}, \underbrace{[1,1], [1,6]}_{[1,\pm 1]}, \underbrace{[3,3], [3,4]}_{[3,\pm 3]}, \underbrace{[4,3], [4,4]}_{[4,\pm 3]}\}$

$E(\mathbb{F}_7)$ is an abelian group of order 9, isomorphic $\mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_9$

Verify that $[4, 4]$ is of order 9,

Elliptic Curves over $\mathbb{F}_p$

- Elliptic Curves over finite fields are useful for cryptographic applications. Elliptic Curves Cryptography ECC
- Consider curves $y^2 = x^3 + Ax + B$, with $A, B \in \mathbb{F}_7$, $\Delta = 4A^3 + 27B^2 \neq 0$ over the field $\mathbb{F}_7 = \{0, 1, 2, 3, 4, 5, 6\}$
- There are at most 49 possible elliptic curves of this form over $\mathbb{F}_7$. (definitely fewer, since Δ can be 0)
- For a fixed such curve, $\exists$ 7 possible values for x and $\forall\, x \,\exists\, 0, 1, 2$ values of y, plus the point at ∞, $\rightsquigarrow |E(\mathbb{F}_7)| \leq 15$

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Verify that $[4,4]$ is of order 9, $\rightsquigarrow E(\mathbb{F}_7) \cong \mathbb{Z}_9$

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- The error term is bounded in absolute value by $2\sqrt{p}$

Useful bibliographical references

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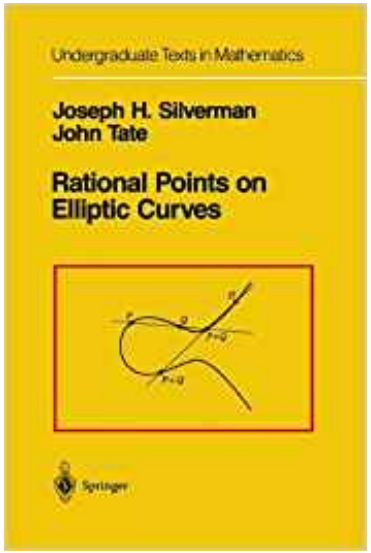

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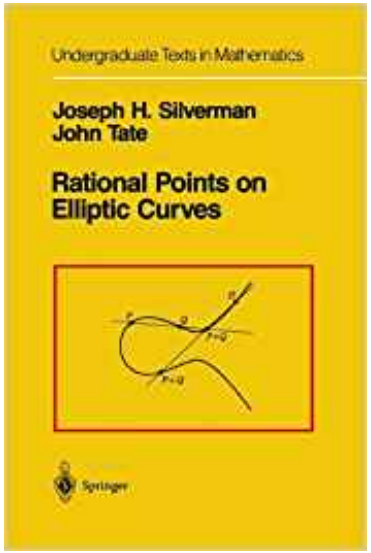

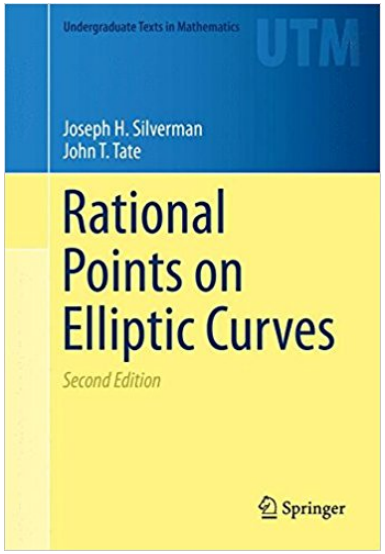

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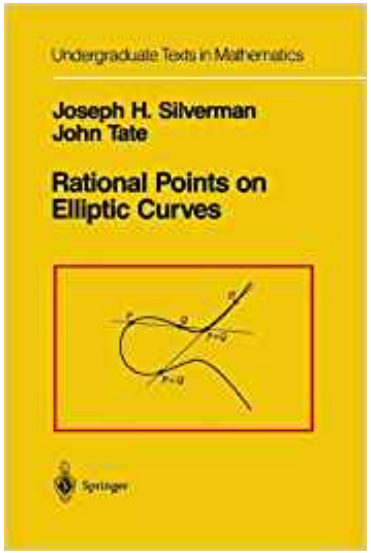

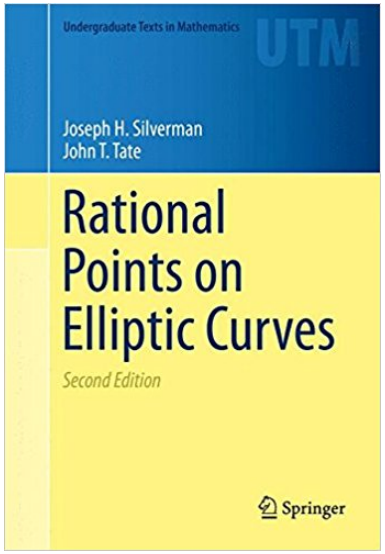

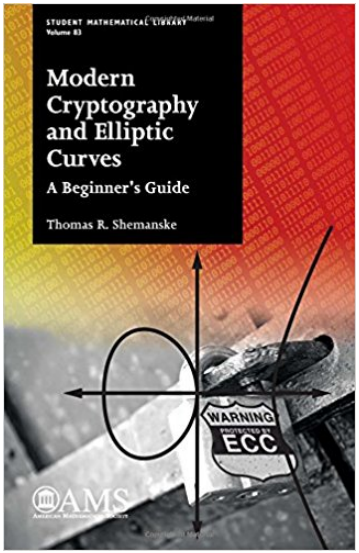

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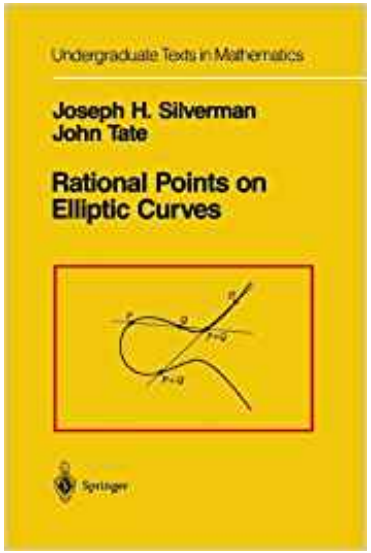

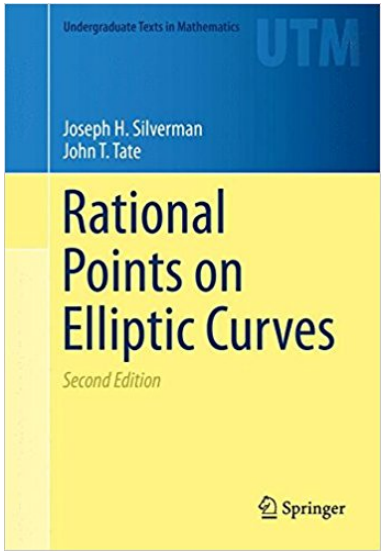

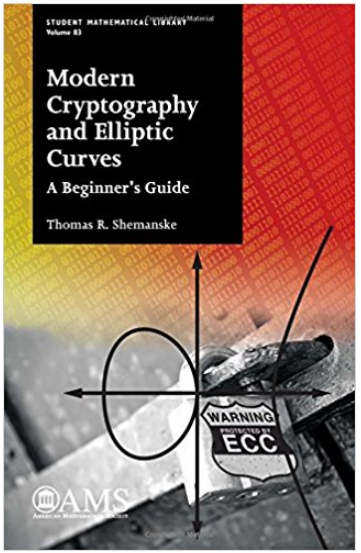

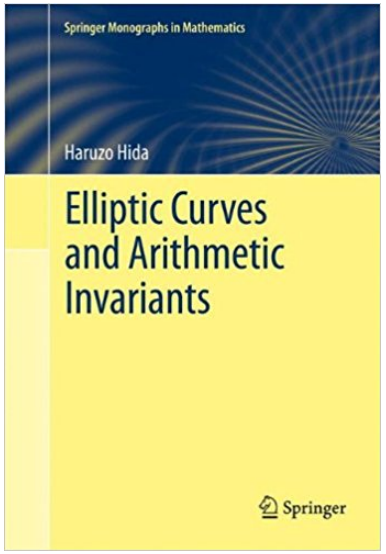

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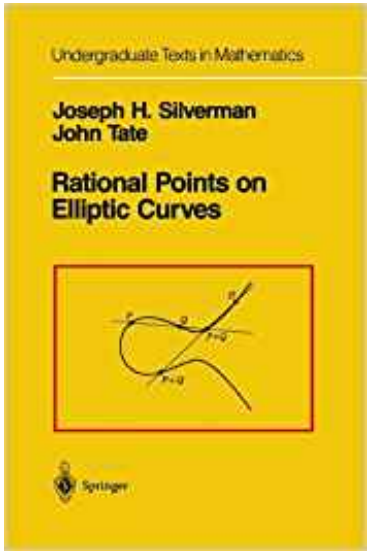

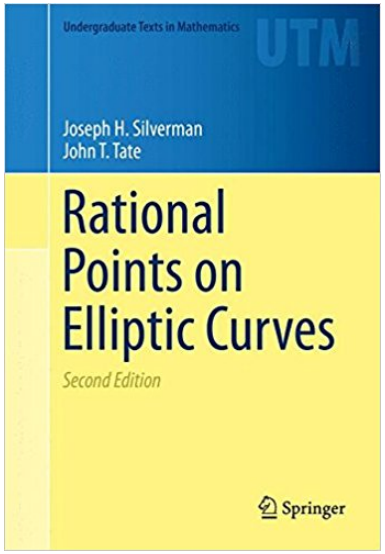

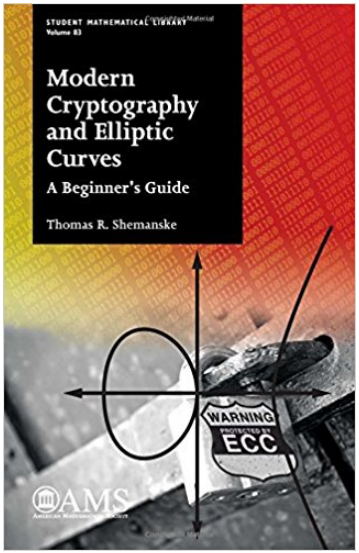

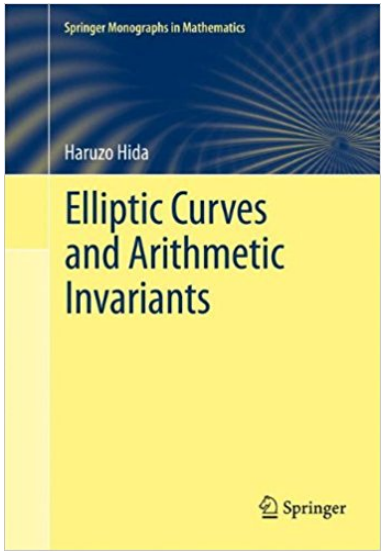

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