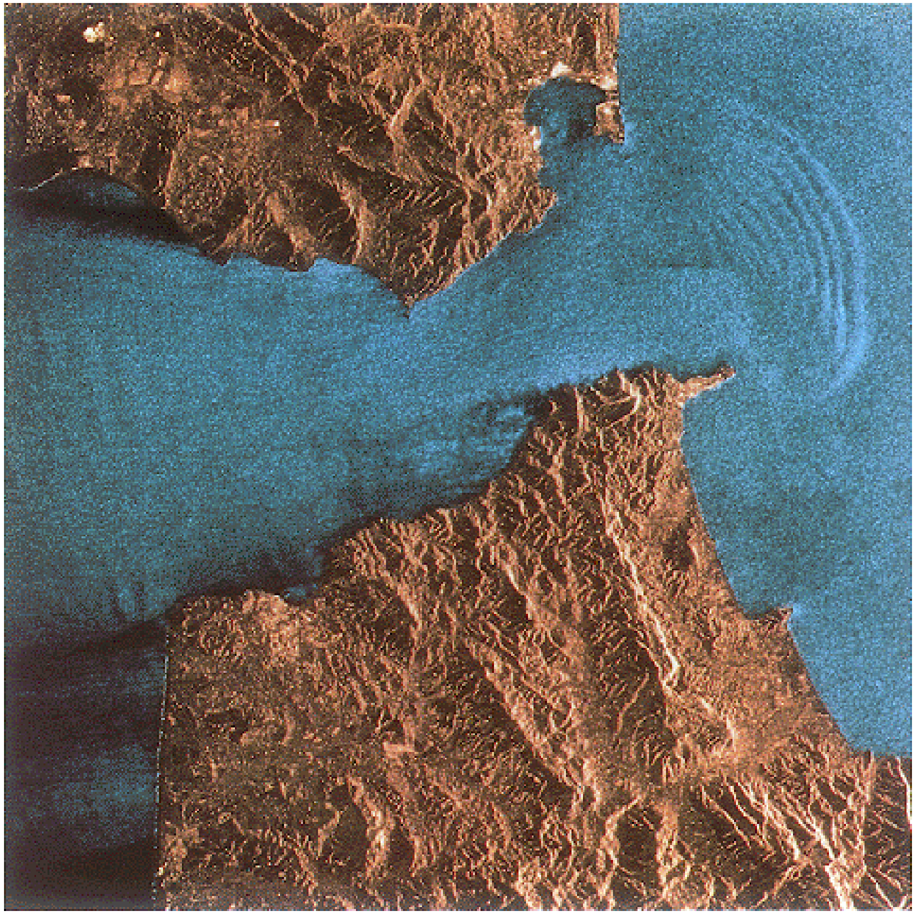
Solitons: extraordinary waves from oceans to cold atoms and beyond



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Outline

- General introduction
  - PDEs, waves, dispersion, nonlinearity, solitons, observations
- Historical timeline
  - Waves – water waves, from Euler to Korteweg & de Vries
- The works of Fermi-Pasta-Ulam (FPU) and Zabusky-Kruskal
  - The FPU "paradox" and solitons
- Solitons in different scales and contexts:
  - Nonlinear Optics - Optical fiber communications
  - Solitons in quantum systems (Bose-Einstein condensation)
  - Solitons in acoustics
  - Blood pressure waves
  - Rogue waves
- Conclusions

Waves

- The study of waves can be traced back to antiquity where philosophers, e.g., Pythagoras (560-480 BC), studied the relation of pitch and length of string in musical instruments.
- Waves are of broad interest – cf. prints by Japanese artist Katsushika Hokusai (1760-1849).

“Fast cargo boat battling waves” (1805)

“The Great Wave of Kanagawa” (1831)

Linear equations and useful notions

- Consider a differentiable scalar function $u(x,t)$, a partial differential operator L, and the PDE: $L[u] = 0$

  Let u_1 and u_2 two different solutions of the PDE;
  the latter is said to be linear iff: $L[u_1+u_2] = L[u_1] + L[u_2] = 0$

- **Dispersive wave equations**: existence of plane waves:

$$u(x,\mathrm{t}) = \exp(\mathrm{i}\theta), \quad \theta = kx\text{-}\omega t, \quad k, \omega \in \mathbb{R}$$

Temporal period: $\mathrm{T}=2\pi/\omega$. Spatial period (wavelength): $\lambda=2\pi/k$

- Dispersion relation: $D(\omega,k) = 0$ or $\omega = \omega(k)$
- Phase velocity: $\mathrm{v_p} = \omega/\mathrm{k}$ - Group velocity: $\mathrm{v_g}=\partial\omega/\partial\mathrm{k} \equiv \omega'(k)$
- If $\omega(k) \in \mathrm{R}$ and $\omega''(k) \neq 0$: the PDE / wave: **Dispersive**

Linear non-dispersive equations

- Simplest linear 1st-order problem: transport (advection) equation

$$u_t + c\,u_x = 0,\qquad u(x,0) = F(x)$$

$\omega = ck \in \mathbb{R}$, $\omega''(k) = 0$

Non-dispersive system

➢ Method of characteristics

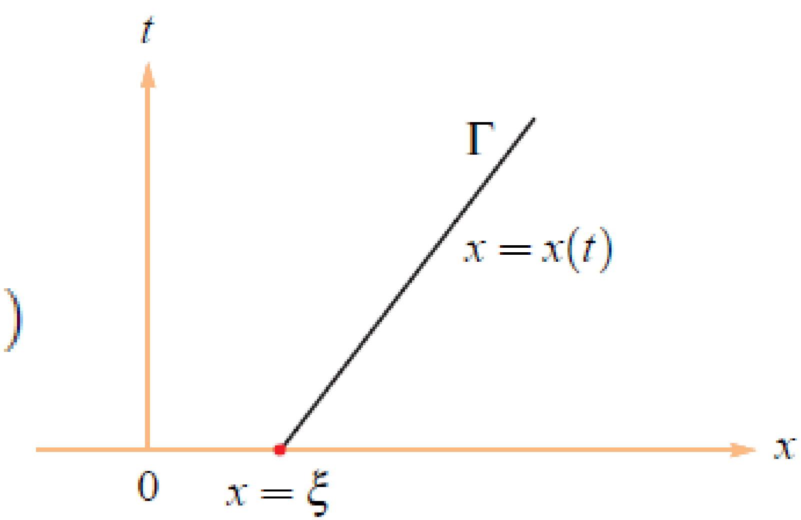
Reduce the problem to an ODE along some curve Γ: $x=x(t)$ such that $du/dt=0$

$$\frac{du(x(t),t)}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t}\frac{dt}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = 0 \qquad \frac{dx}{dt} = c$$

$$\Rightarrow \frac{dx}{dt} = c \;\Rightarrow\; x(t) = c\,t + \xi$$

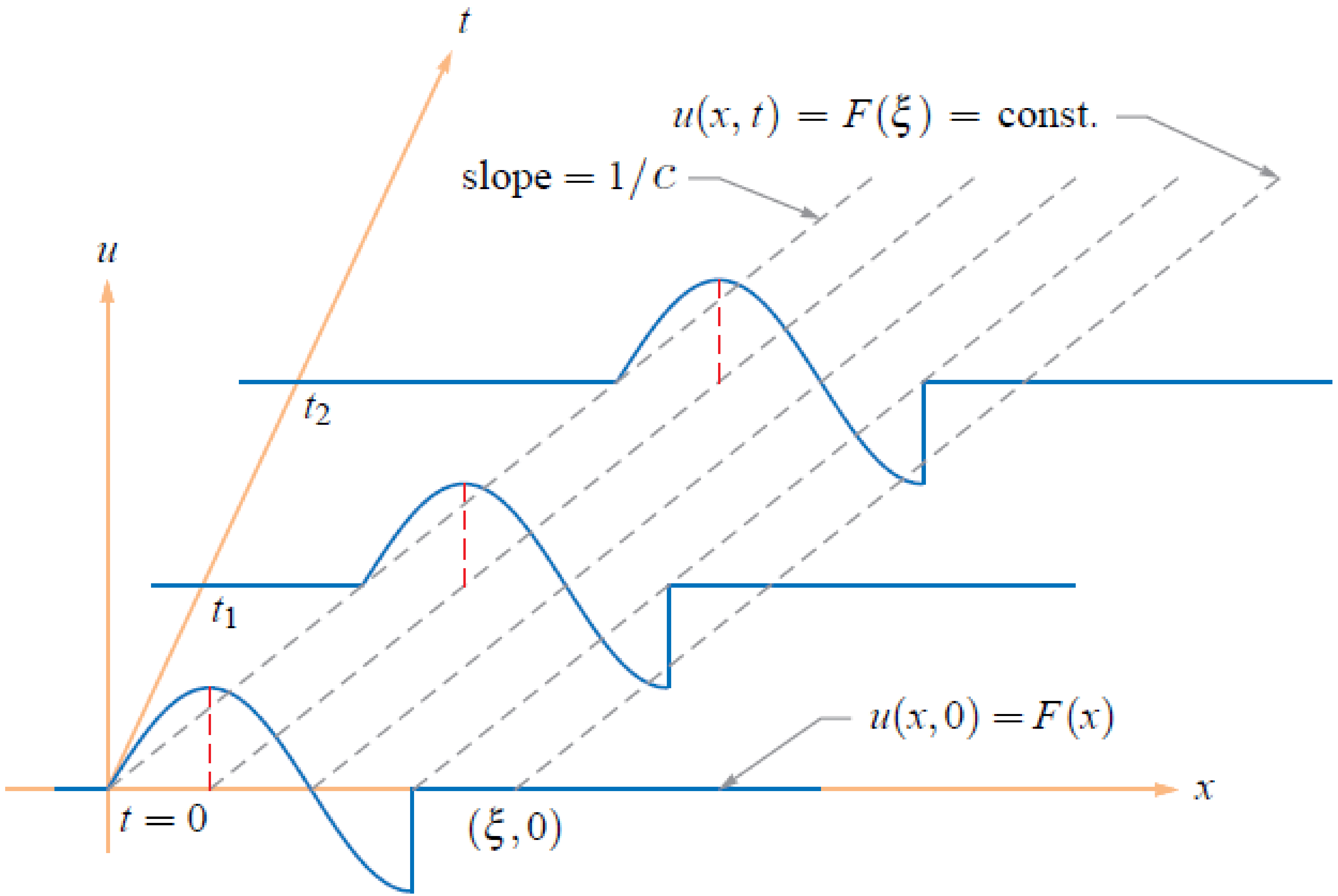
$$\Rightarrow \frac{du}{dt} = 0 \;\Rightarrow\; u(x,t) = u(\xi,0) = F(\xi)$$

General solution: $u(x,t) = F(x - c\,t)$



Solution of the transport equation

The IC, $F(x)$, is simply translated **without changing shape**



The effect of dispersion

- A simple dispersive model: **linearized KdV equation**

$$u_t + u_{xxx} = 0$$
$$u(x,0) = u_0(x)$$

ω=-$k^3 \in \mathbb{R}$, $\omega''(k)$ =-6$k \neq 0$

Dispersive system

➢ **Fourier transform method:**
Use Fourier transform (FT) to derive the ODE: $U_t + ik^3U = 0$; then, solve the ODE, and find the inverse FT:

$$u(x,t) = \int_{-\infty}^{\infty} A(k) \exp[i(kx + \underbrace{k^3}_{-\omega(k)} t)] \frac{dk}{2\pi} \qquad A(k) = \int_{-\infty}^{\infty} u_0(x')e^{-ikx'}\, dx'$$

- Superposition of elementary waves ~ exp[ikx-i$\omega(k)t$]
- Different components propagate with different speeds
- ➡ **the wave profile changes shape and spreads out – disperse**
- ➡ **Need for asymptotics for integrals… ☹**

An example: decay of a pulse

- Express the solution of the linearized KdV equation as:

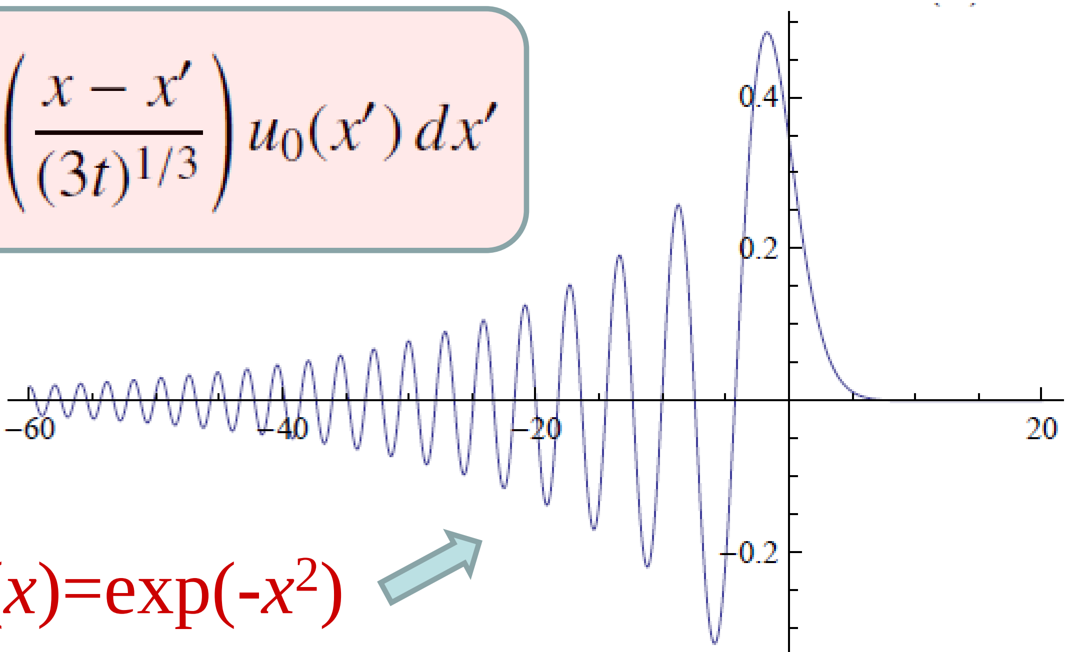
$$u(x,t) = \int_{-\infty}^{\infty} u_0(x')G(x-x',t)\,dx',$$

$$G(x,t) = \frac{1}{\pi}\int_0^{\infty} \cos(kx + k^3 t)\,dk.$$

Airy function

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^{\infty} \cos\left(\frac{1}{3}k^3 + xk\right) dk,$$

$$u(x,t) = \frac{1}{(3t)^{1/3}} \int \mathrm{Ai}\left(\frac{x-x'}{(3t)^{1/3}}\right) u_0(x')\,dx'$$



- Numerical solution for $u_0(x)=\exp(-x^2)$

The effect of nonlinearity

- Simplest nonlinear 1st-order problem:
  Hopf (inviscid Burgers) equation

$$u_t + u u_x = 0$$
$$u(x,0) = f(x)$$

"nonlinear transport equation": here $c \to c(u) = u$

➢ **Method of characteristics:**

Again, $du/dt = 0$ along the curves $dx/dt = u \Rightarrow \boxed{u = f(x - ut)}$

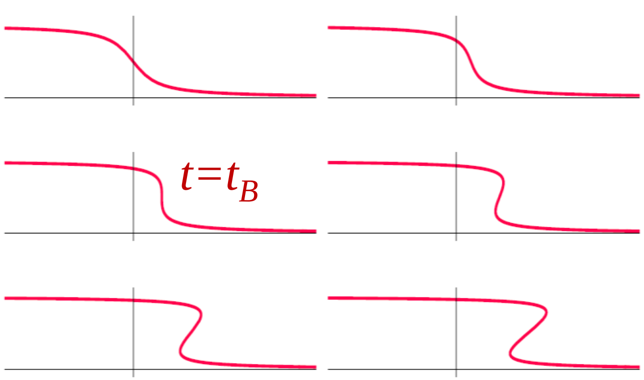
Alternatively: $u(x,t) = f(\xi)$ along $x = \xi + f(\xi)t$

➡ The solution ceases to exist (blows up) at finite time for $f'(\xi) < 0$

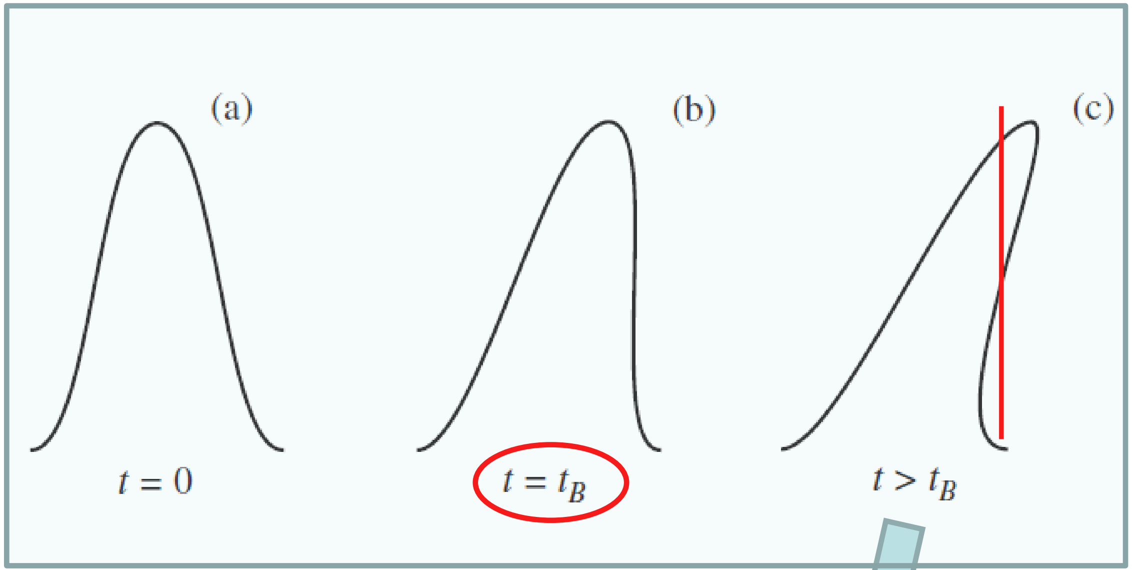
$$\left.\begin{aligned} u_x &= u'(\xi)\xi_x \\ \xi_x &= \frac{1}{f'(\xi)t + 1} \end{aligned}\right\}$$

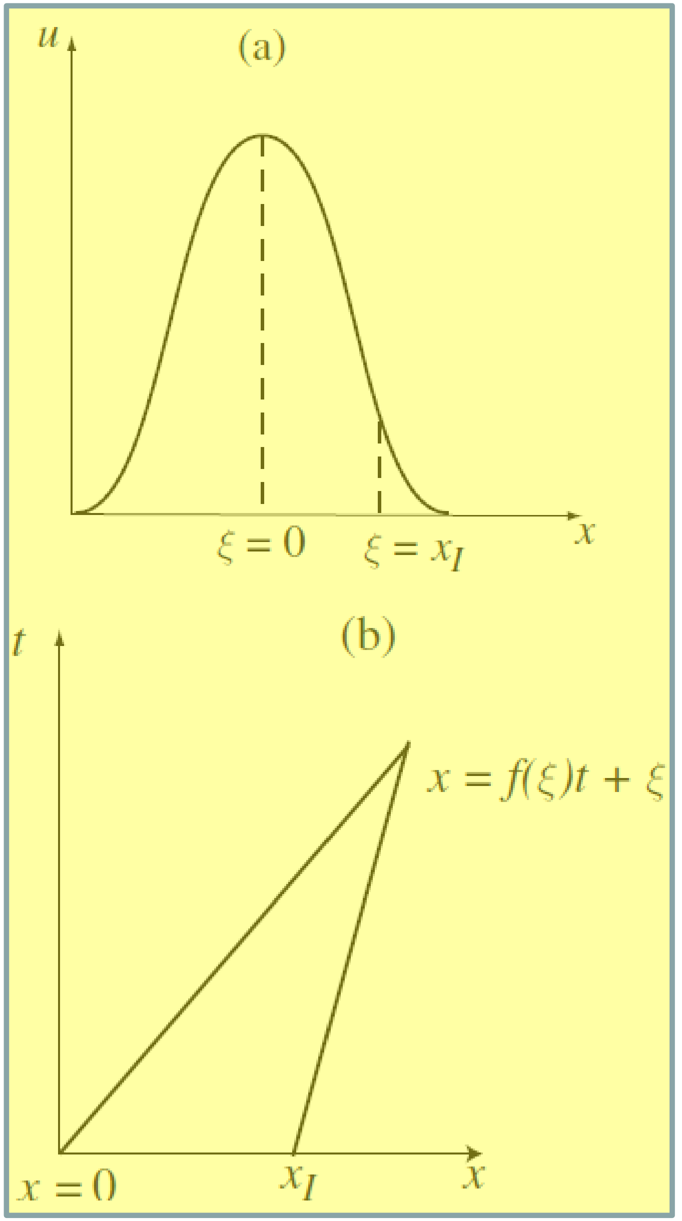
Breaking time t_B:

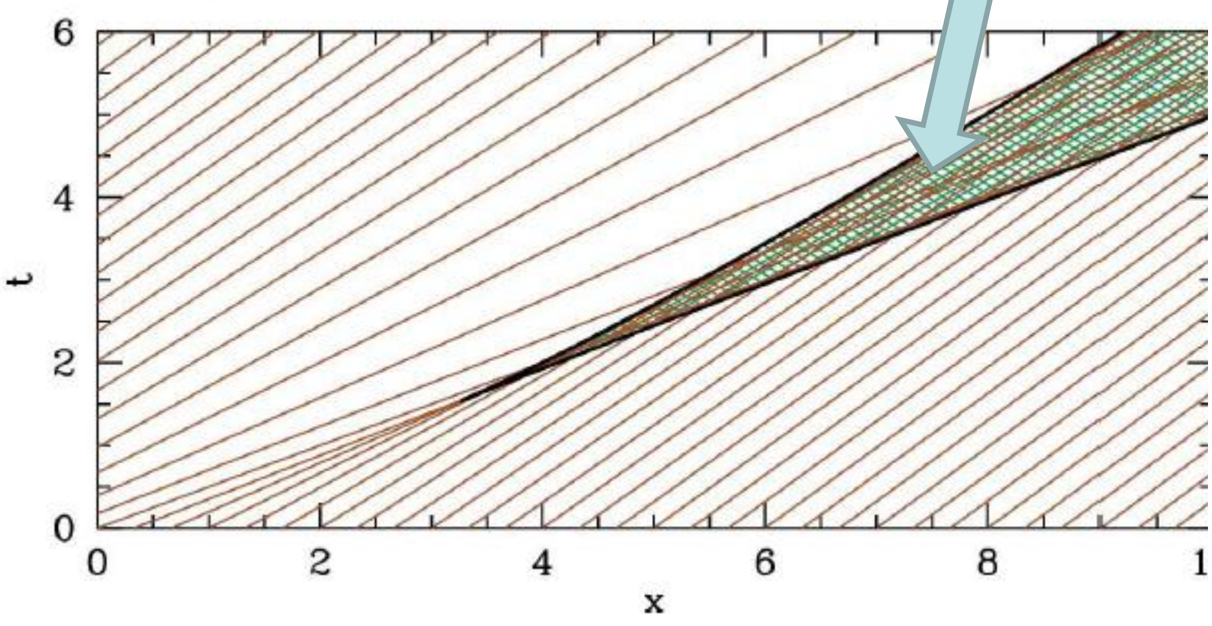
$$t = t_B = 1/\max(-f')$$



Shock formation – hump-like initial data

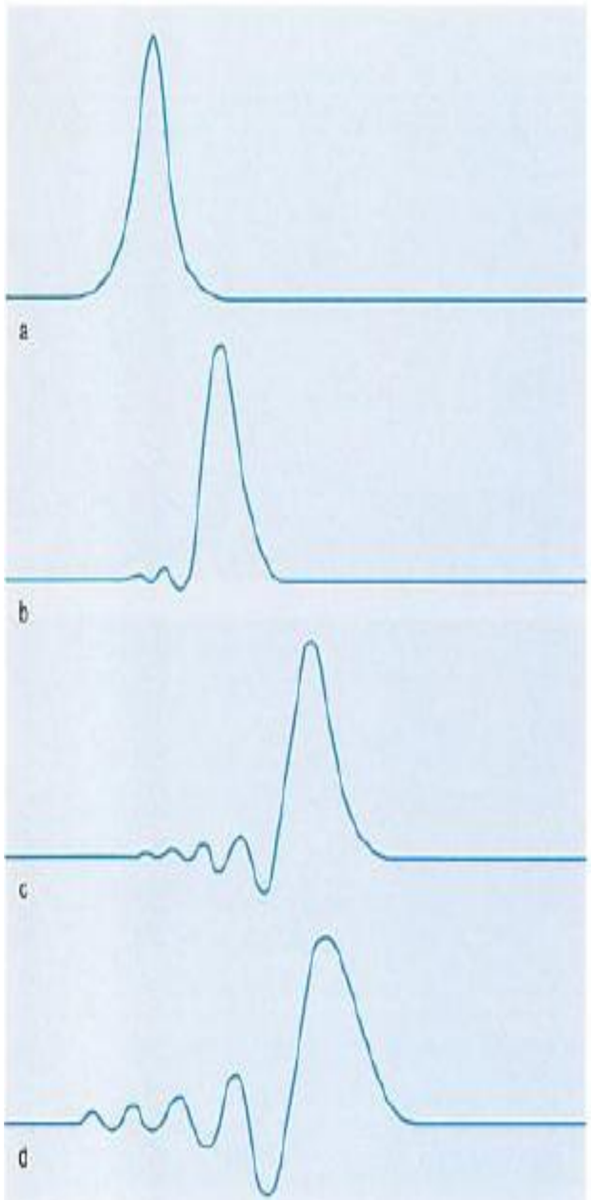


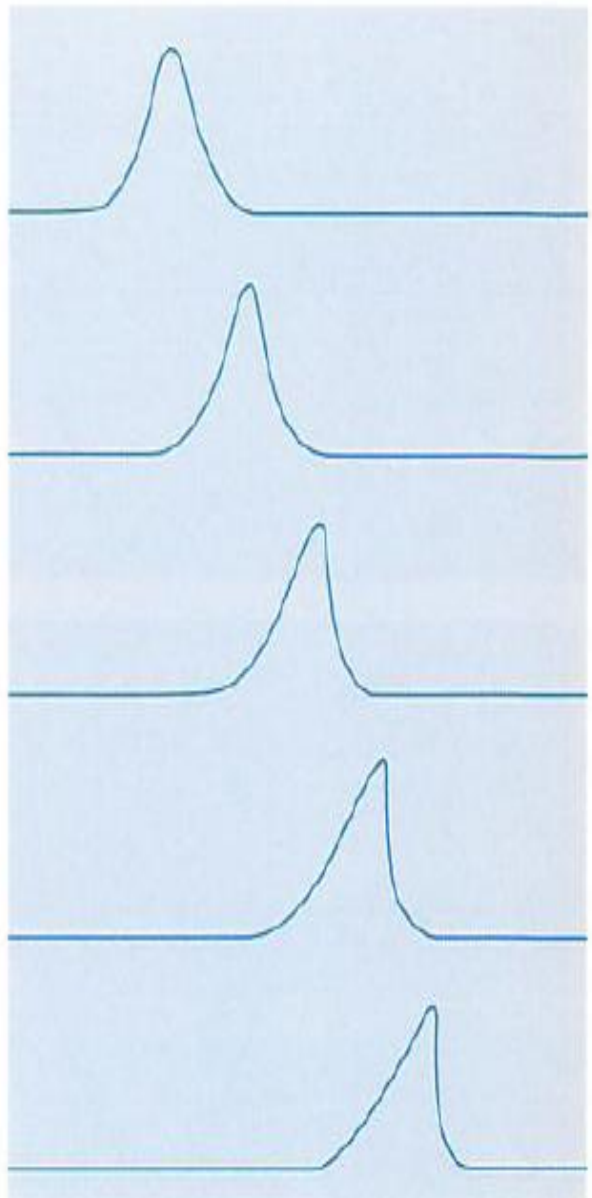




Shock formation: Art vs Nature

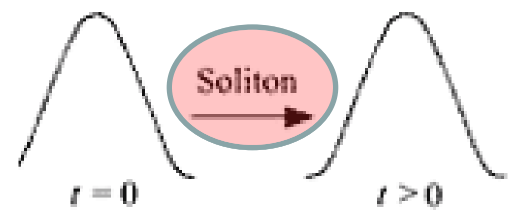
Dispersion vs nonlinearity





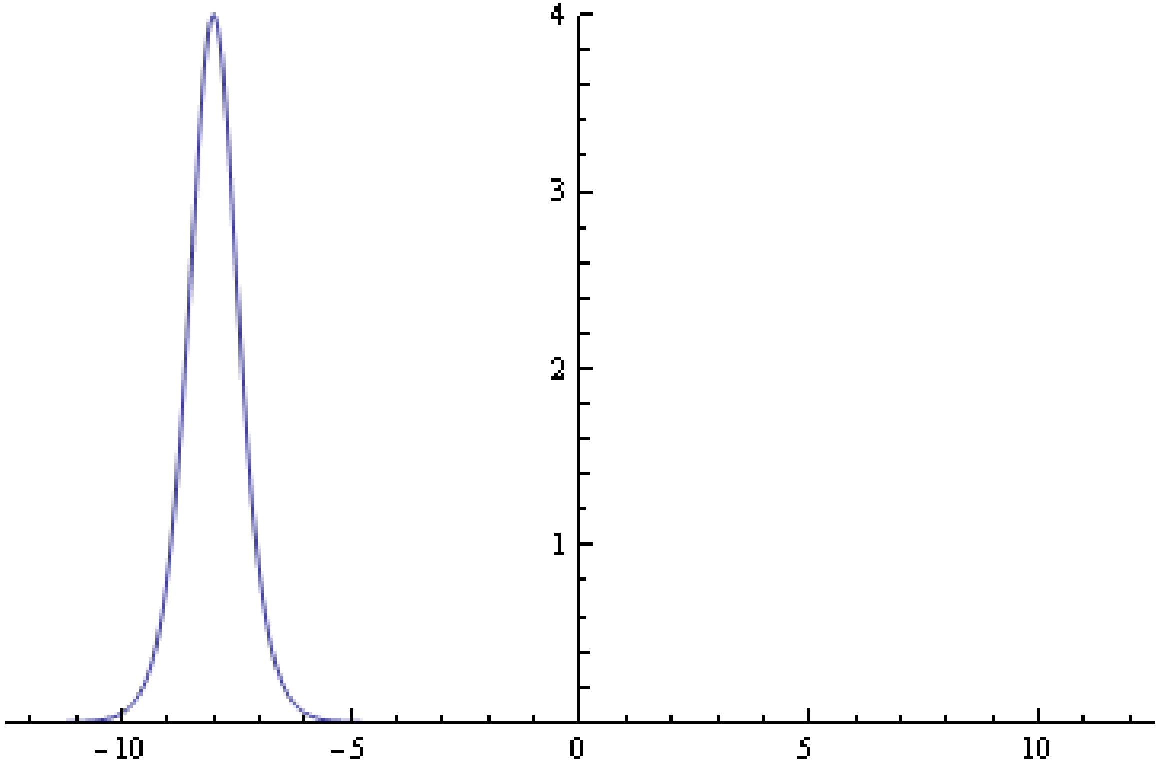
When dispersion and nonlinearity are *counterbalanced*

$\Rightarrow$ **solitons!**



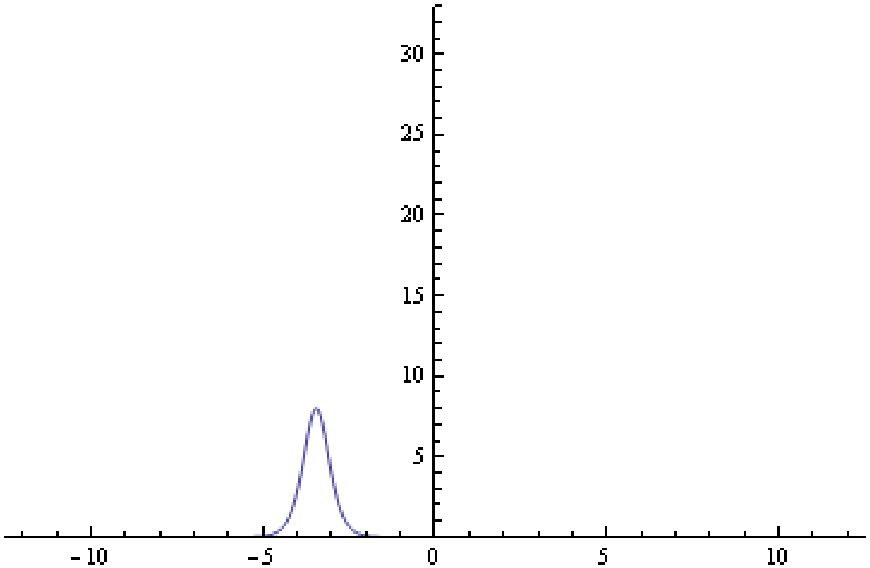
What is a soliton?

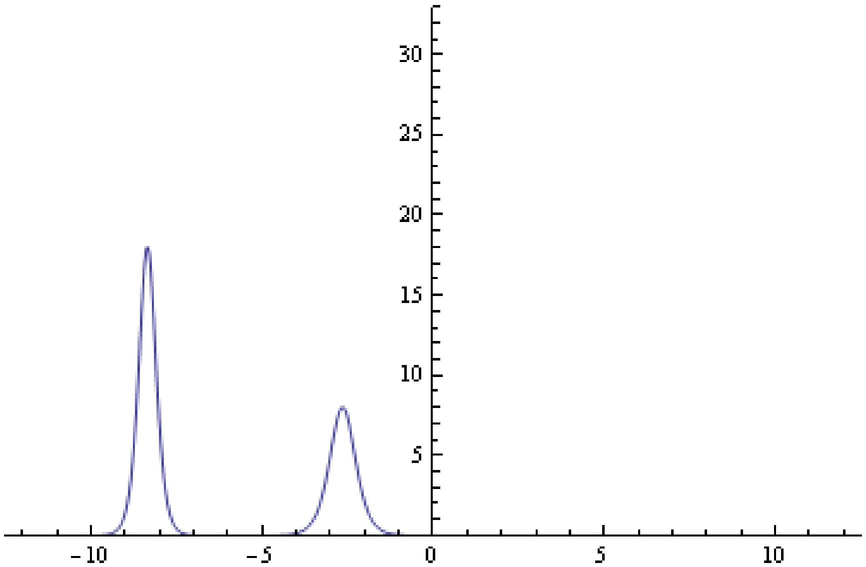
- Stable **localized** (solitary) waves propagating undistorted in **nonlinear dispersive media**



Soliton collisions

- Solitons collide with other solitons without any change in their shapes (particle properties!)





Two-soliton collisions **Three-soliton collisions**

- Soliton amplitude ~ soliton velocity

➡ **Larger solitons travel faster that smaller ones**

Soliton Equations

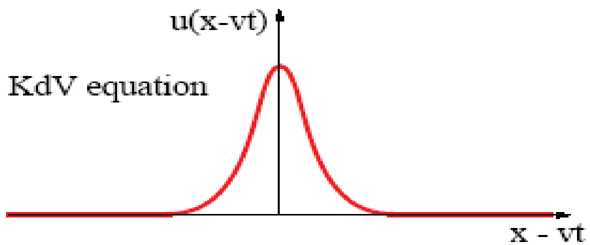
- **Nonlinear Dispersive PDEs** (infinite-dimensional dynamical systems)
- <u>Ideally</u>: **Completely integrable** (infinite number of integrals of motion)

Examples

- **Korteweg-deVries (KdV)**

  (shallow water waves, plasmas)

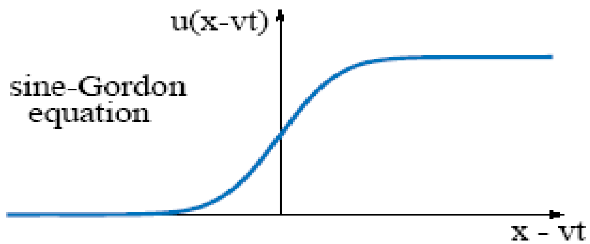
  $$u_t + 6uu_x + u_{xxx} = 0$$



- **Sine-Gordon (SG)**

  (field theory, Josephson junctions)

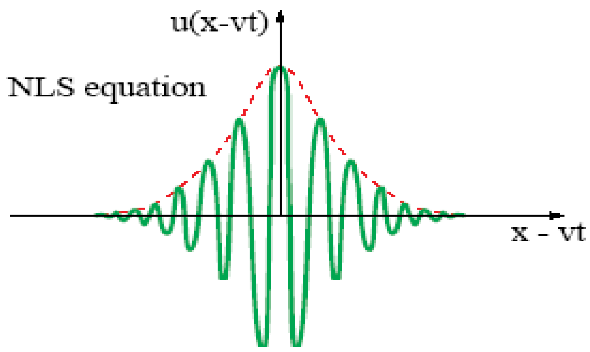
  $$u_{xx} - u_{tt} = \sin u$$



- **Nonlinear Schrödinger (NLS)**

  (deep water waves, optics, BECs)

  $$iu_t + \frac{1}{2}u_{xx} + |u|^2 u = 0$$



Solitons & periodic waves of the KdV

- KdV equation: $u_T + uu_X + \delta^2 u_{XXX} = 0$
- Traveling waves: $u = U(\zeta), \; \zeta = (X - CT - X_0)$

$$-CU_\zeta + UU_\zeta + \delta^2 U_{\zeta\zeta\zeta} = 0.$$

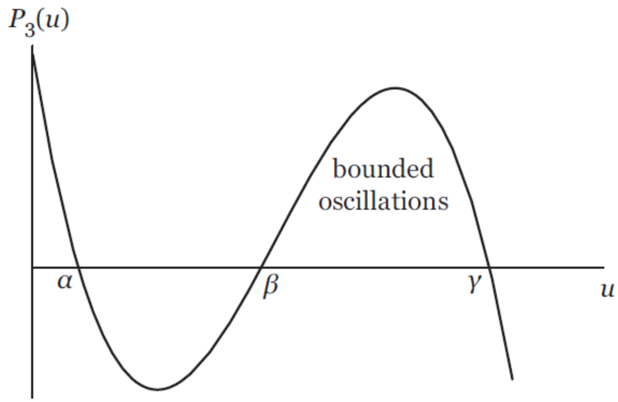
- Boundary conditions: $U \to U_\infty$ as $|\zeta| \to \infty$

➢ Integrate once: $\delta^2 U_{\zeta\zeta} + \frac{U^2}{2} - CU = \frac{E_1}{6}$

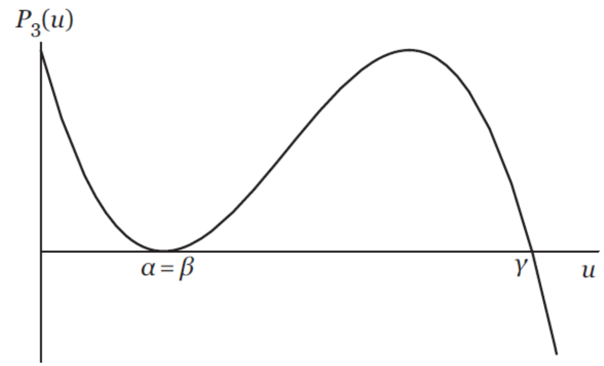
➢ Multiply by U_ζ & integrate again: $\frac{\delta^2}{2} U_\zeta^2 + \frac{U^3}{6} - C\frac{U^2}{2} = \frac{E_1}{6} U + \frac{E_2}{6}$

$$\frac{\delta^2}{2} U_\zeta^2 = \frac{1}{6} P_3(U) \equiv -V(\zeta), \quad P_3(U) = -U^3 + 3CU^2 + E_1 U + E_2$$

Forms of the inverted potential & solutions



Three different real roots, α, β, γ



The case of a double root $\alpha = \beta$

Periodic solutions:
cnoidal waves

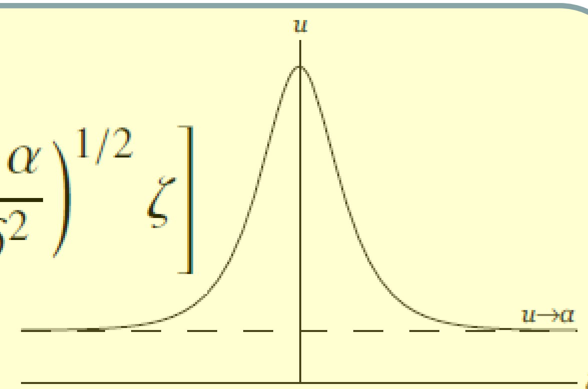
$$U(\zeta) = \beta + (\gamma - \beta) cn^2 \left[\left(\frac{\gamma - \alpha}{12\delta^2} \right)^{1/2} \zeta ; m \right] \quad m = \frac{\gamma - \beta}{\gamma - \alpha}$$

Limiting cases of $\mathrm{cn}(x,m)$: $m \to 0$: $\mathrm{cn}(x,0) = \cos x$; $m \to 1$: $\mathrm{cn}(x,1) = \mathrm{sech}\, x$

Special case:
$\beta \to \alpha \ \Rightarrow m \to 1$
soliton

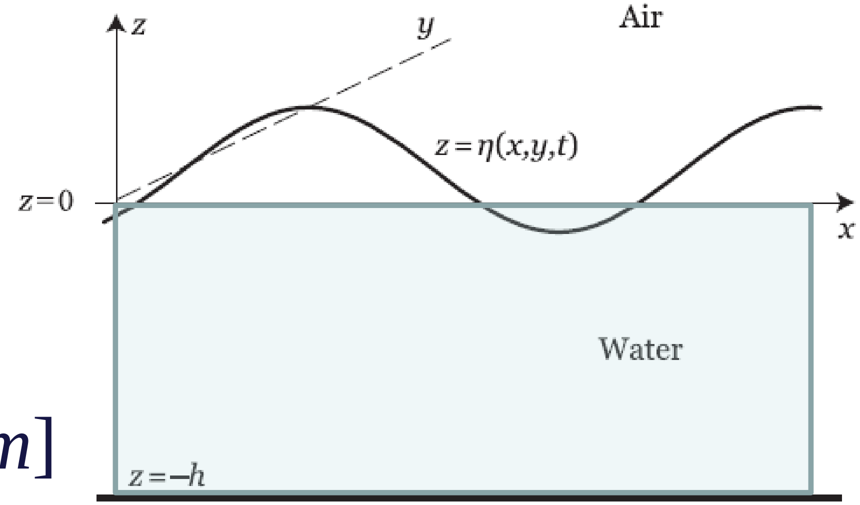
$$U(\zeta) = \alpha + (\gamma - \alpha)\, \mathrm{sech}^2 \left[\left(\frac{\gamma - \alpha}{12\delta^2} \right)^{1/2} \zeta \right]$$

$$\zeta = (X - CT - X_0)$$



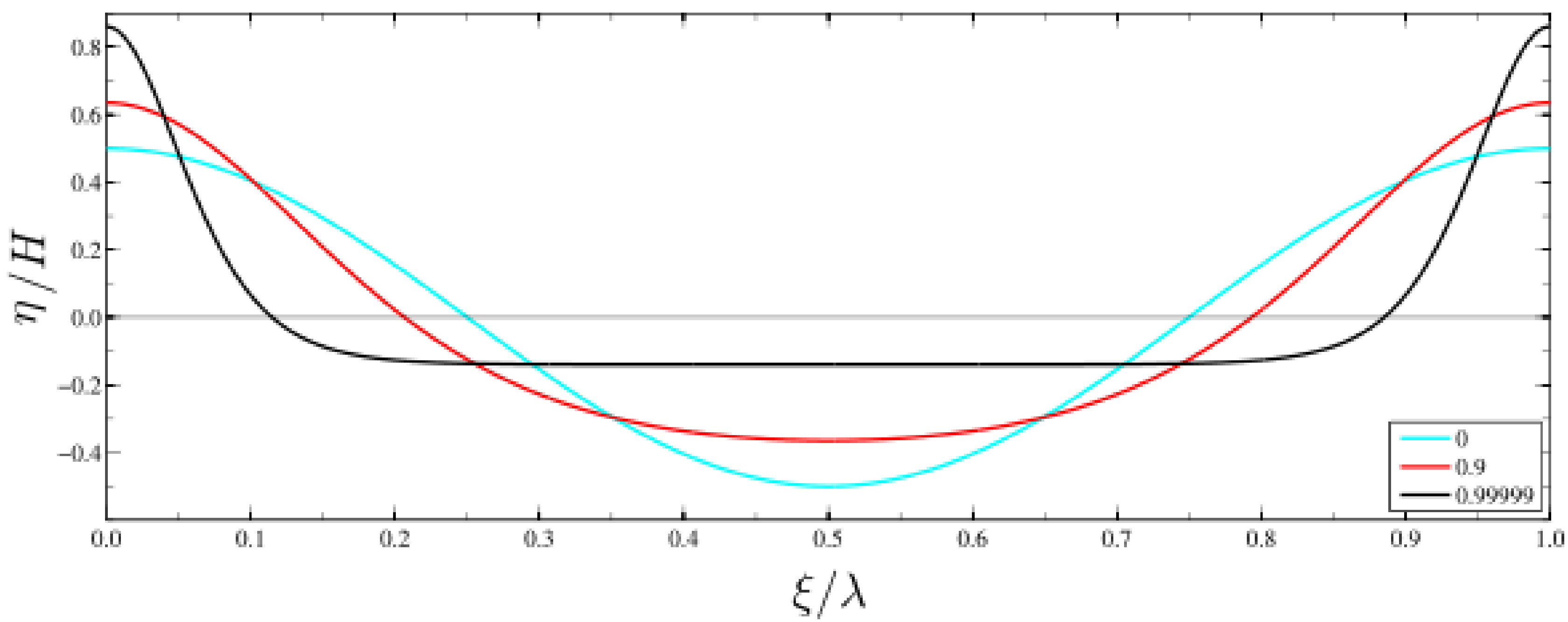
Cnoidal waves & solitons

Water waves: the KdV equation approximately describes the wave elevation $\eta=\eta(x,t)$ above mean height h



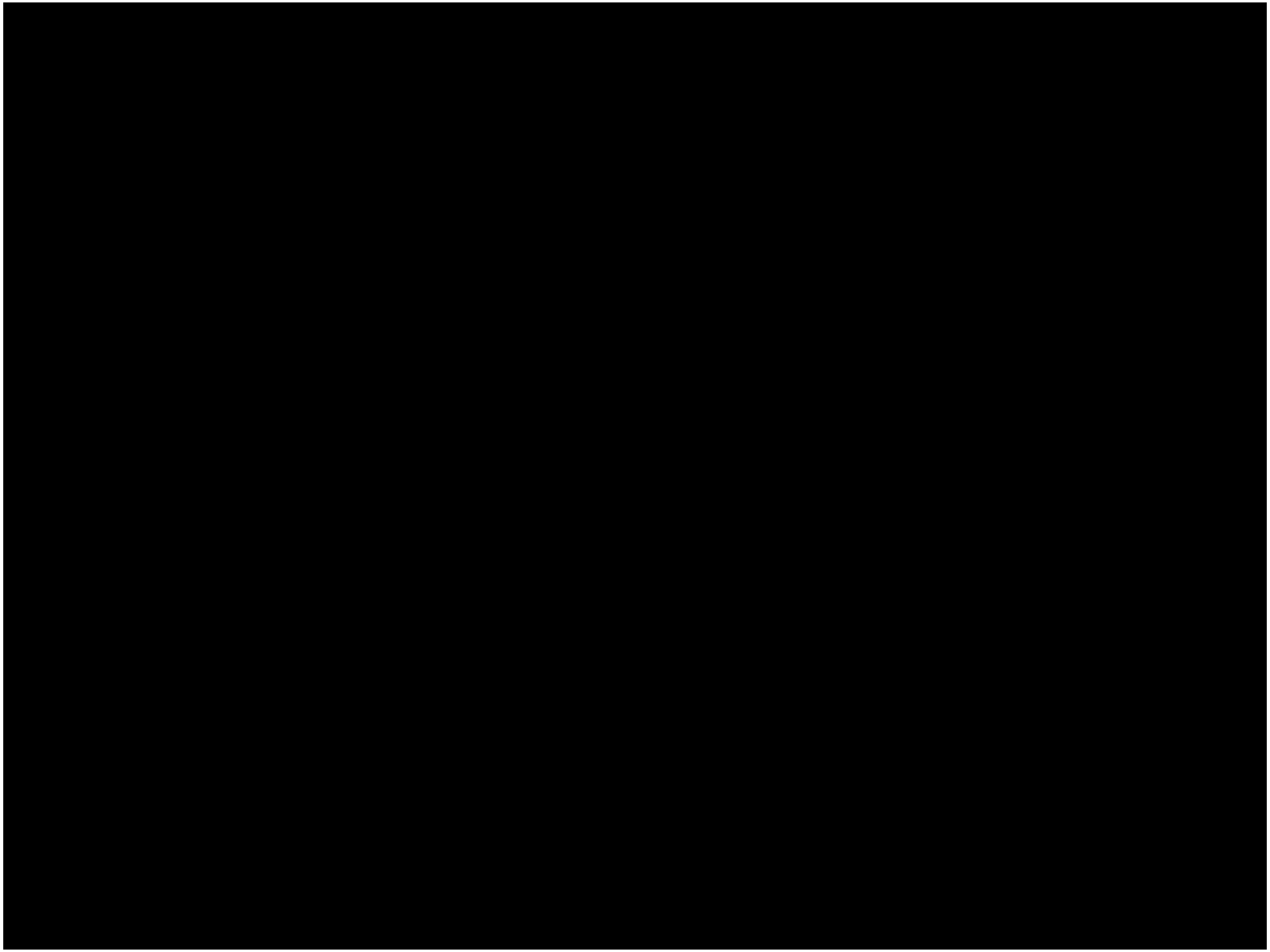
cnoidal wave

$$\eta(x,t)=\eta_0+h\ \mathrm{cn}^2[2K(m)(x\text{-}ct/\lambda),\ m]$$

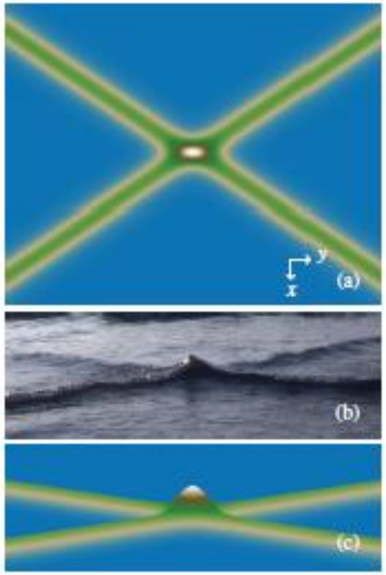


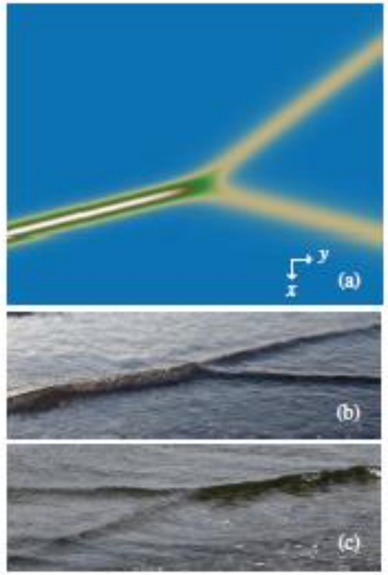
Physical periodic waves and KdV solitons

Soliton interactions in shallow water



Soliton interactions – the KP model

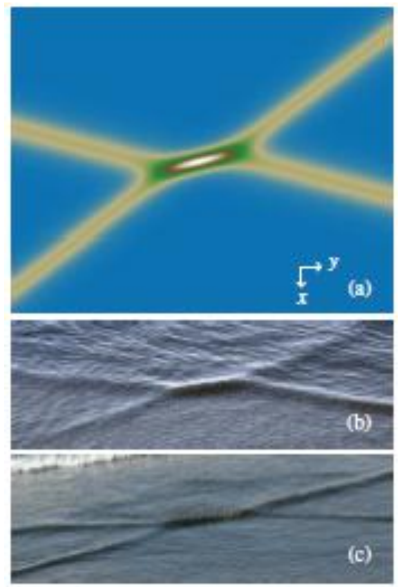


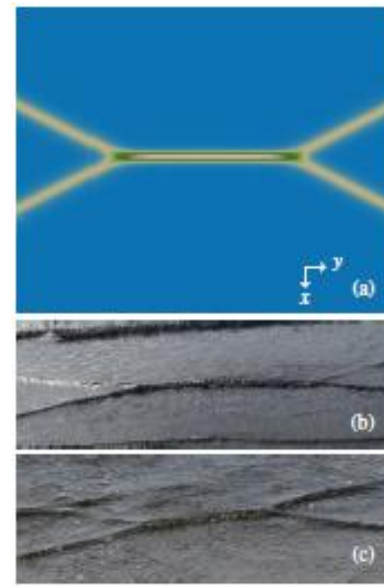


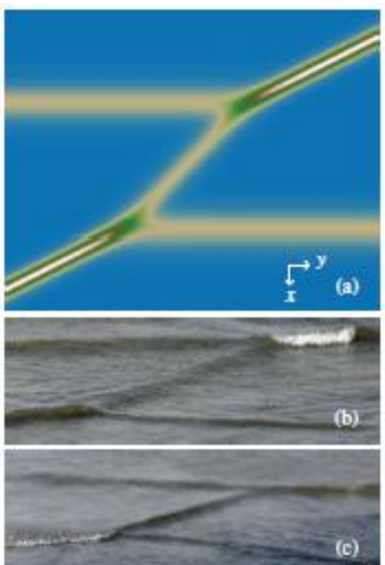
Generalization of KdV in 2D:

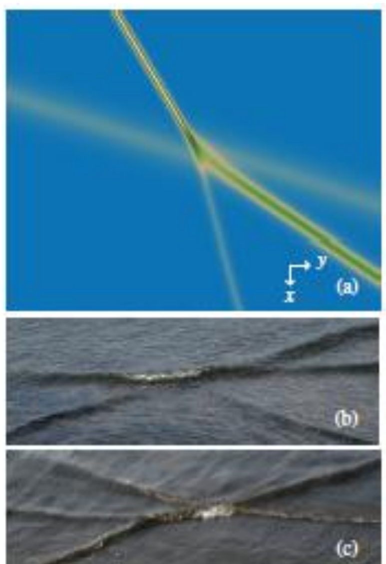
Kadomtsev - Petviashvilli (KP) equation

$$(u_t + 6uu_x + u_{xxx})_x + 3\sigma u_{yy} = 0$$









M. J. Ablowitz and D. E. Baldwin, PRE 2012

… and still another beautiful example!

Localized waves and solitons: can we see them?

- **Localized (solitary) waves and solitons arise in:**

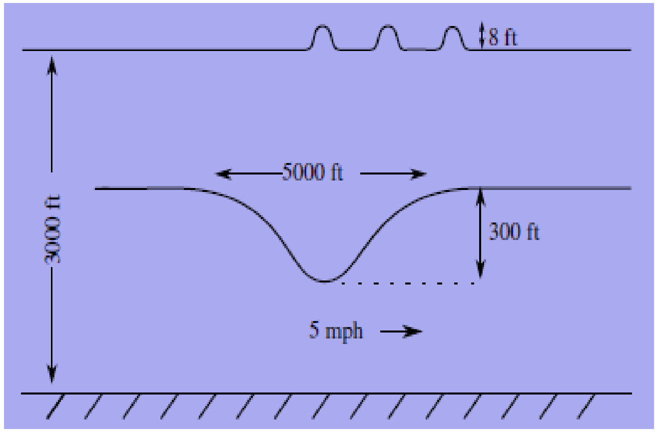
  Water waves, nonlinear optics, Bose-Einstein condensation (BEC), metamaterials, plasmas, solid-state physics, field theory, elasticity, magnetics, biological dynamics, complex systems, acoustics...

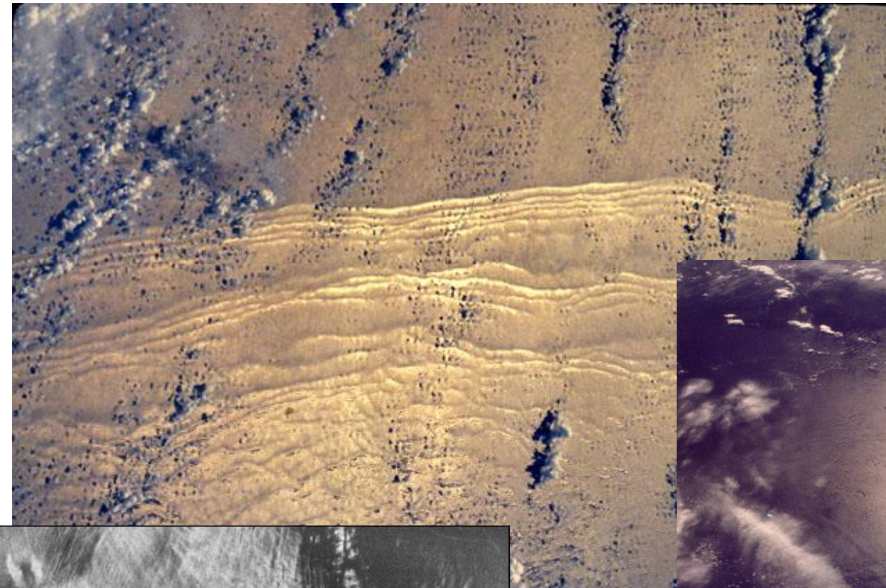
- In many physically relevant cases, such nonlinear waves are very **robust** (mathematically speaking: **stable - persistent**)

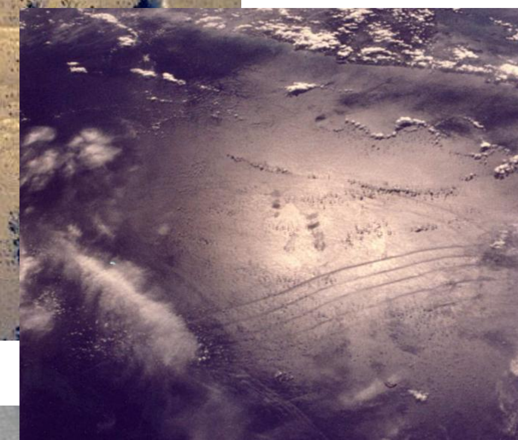
- **Solitons can be used to understand energy localization and the emergence of robust coherent structures in nature**

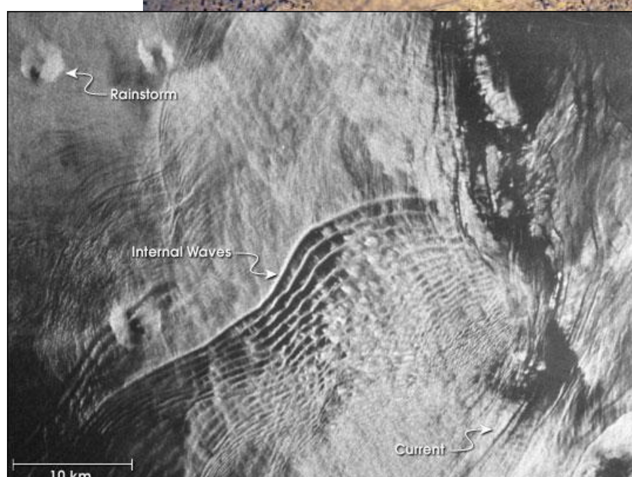
- So, the answer is **yes**! We can observe – and also exploit them!

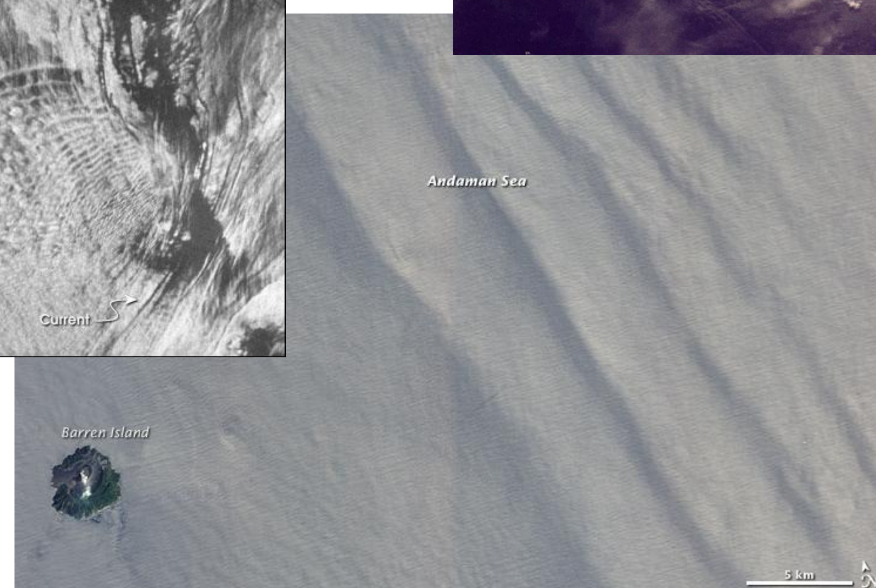
Internal ocean waves











Surface waves of height 6–8 ft in groups of 4–8 every 12 hr 26 min

Waves are on the order of 120 miles in length

The ripples/internal waves persist for 250 miles, or over 2 days.

Internal ocean waves appear to be close to KdV solitons!

Morning Glory

- **Long, roll-shaped clouds**; sometimes can exceed length of **1000 km** and can move at up to **70 km/h!**

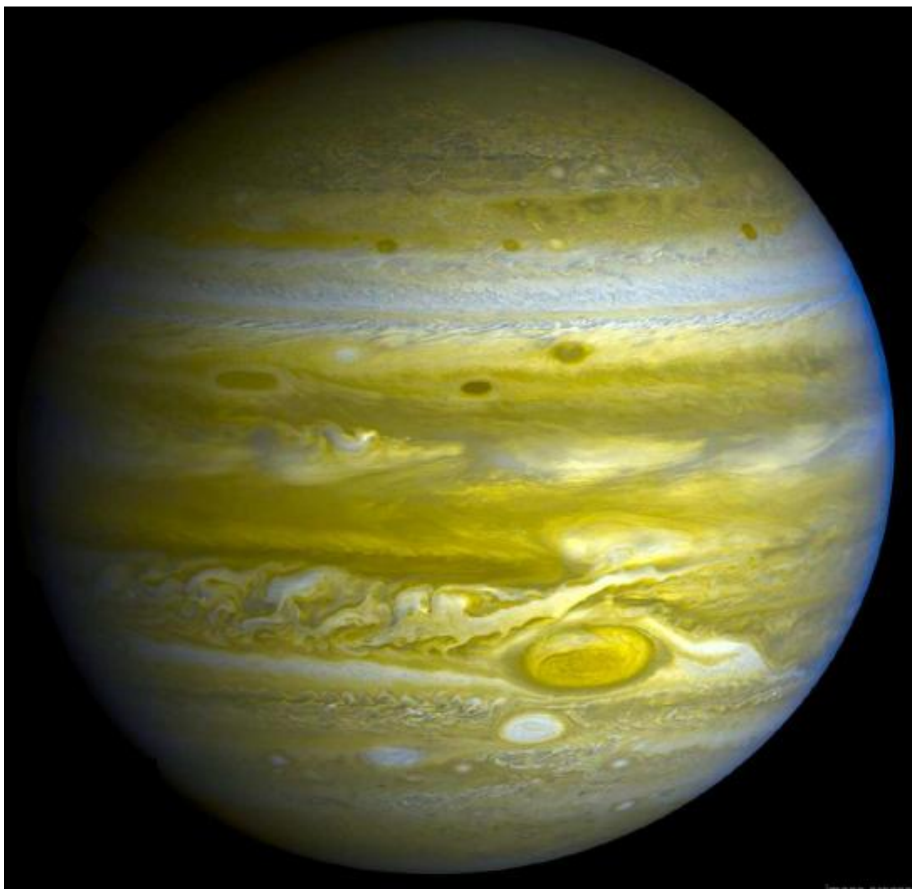
- Morning Glory of the Gulf of Carpentaria in Northern Australia arrives regularly each spring

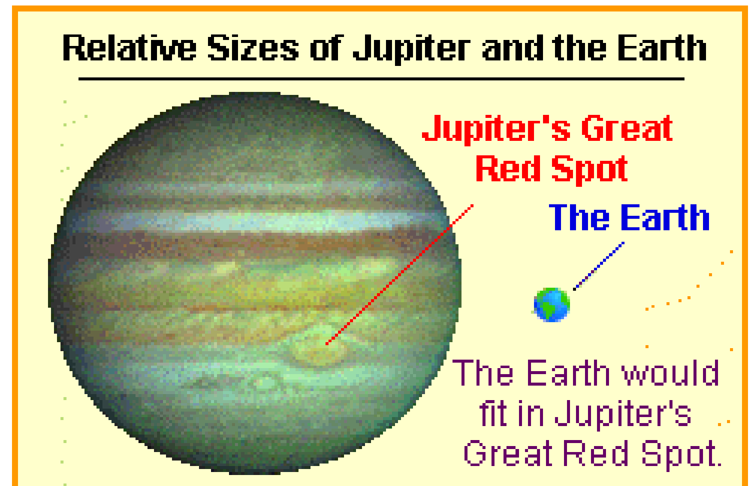
Morning Glory video

Great red spot of Jupiter

- Biggest storm in the solar system → **stable localized wave**: it has been discovered on 1664 and it is still there!

- It is about 28,000 km long and 14,000 km wide: the "spot" is larger than the earth, more than 2 times as large!





Some history

- Solitary waves/solitons of the type we discuss first discovered in **water waves** – **historical timeline follows**

1757 – Fundamental equations of (inviscid) fluids:
Leonard Euler (Swiss)

- Euler was blind for last 20 years of his life
- He wrote almost **900 papers/books** – many when he had little or no vision!
- He is known for many contributions in Mathematics and Physics

Leonard Euler

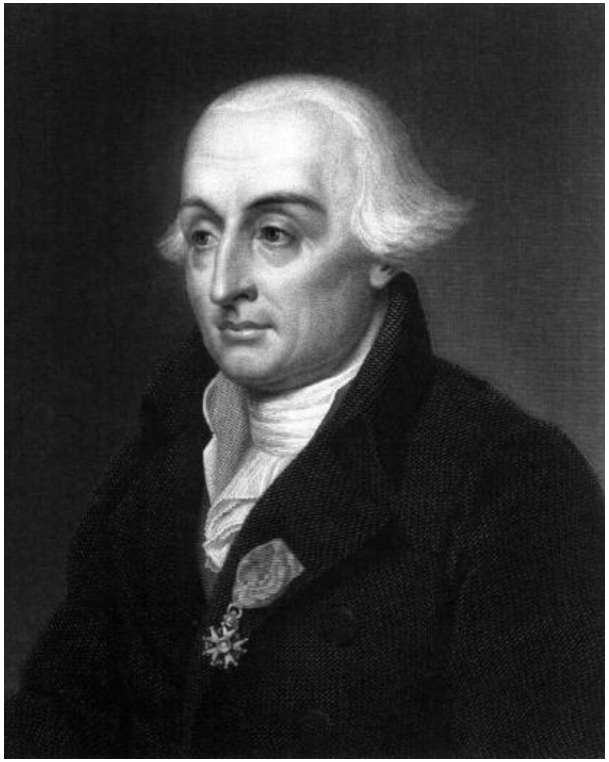
Some history - continued

1776, 1781 – Water waves – small amplitude shallow water – linear wave equation

Pierre Laplace (French) - Laplace equation

Joseph Lagrange (French) - founded Ecole Polytechnique

Pierre Laplace



Joseph Lagrange

Some history - continued

1813 - French Academy of Sciences announced prize competition - **propagation of water waves**

1816 - Augustin Cauchy (French) awarded the prize: linear equation (initial value problem); **Simeon Poisson (French)**, a judge of the committee, also submitted an important paper

Cauchy's work was eventually published in **1827**; (Poisson's work published earlier); closely related to work by **Jean Fourier (French)**

Augustin Cauchy

Simeon Poisson

Jean Fourier

Some history - continued

1837- British Association for the Advancement of Science (BAAS) sets up a "Committee on Waves"; one of two members was **John Scott Russell** (1808-1882; **Scottish** Naval Scientist). Russell was interested in the efficient design of boats.

1837, 1840, 1844

Russell's major effort:

"Report on Waves" to the BAAS

-describes a remarkable discovery:

The Wave of Translation *

John Scott Russell

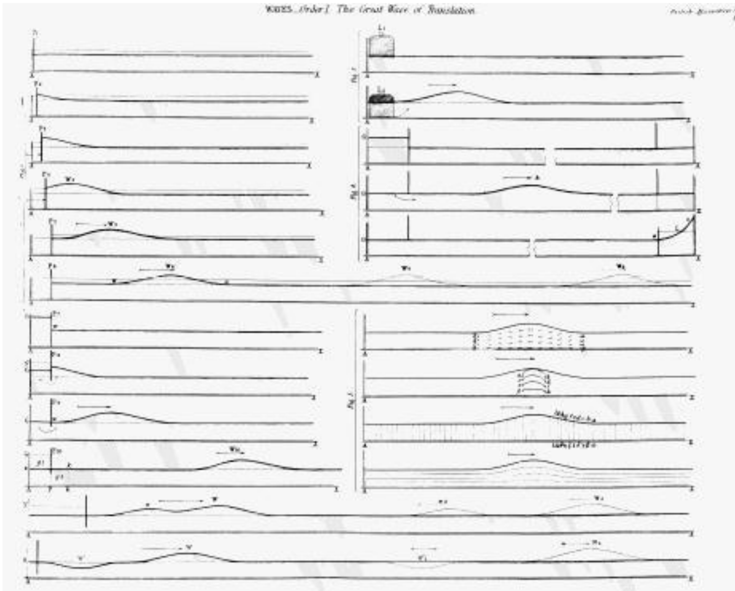
* **"Report on Waves"**: Report of the fourteenth meeting of the British Association for the Advancement of Science, York, September 1844 (London 1845), pp 311-390, Plates XLVII-LVII.

Russell's "Wave of Translation"

"I was observing the motion of a boat drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel. It accumulated round the prow of the vessel in a state of violent agitation, then suddenly ... it rolled forward with great velocity, assuming the form of a large **solitary** elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently **without change of form** or diminution of speed.

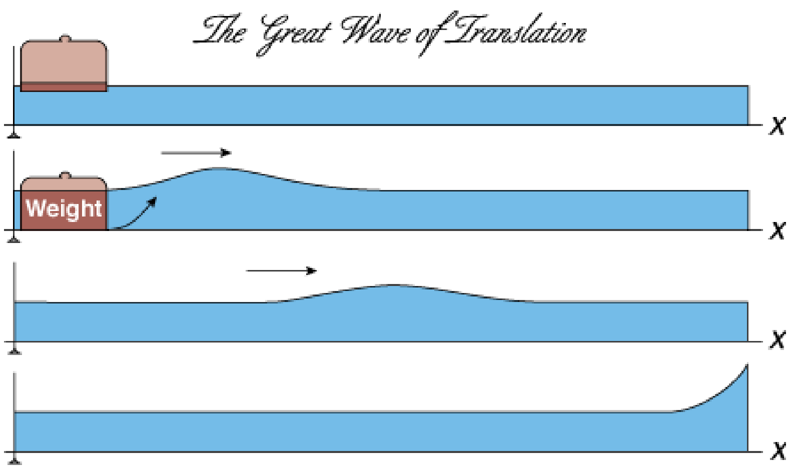
I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure ... a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of **August 1834**, was my first chance interview with that **singular and beautiful phenomenon**..."

“Wave of Translation”: 1834-today

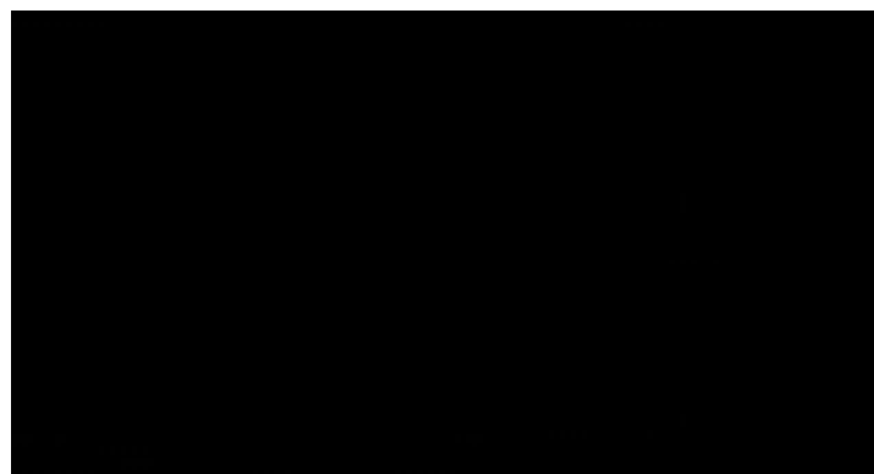


John Scott Russell’s report: experiments

1995, Union Canal, Heriot-Watt University



Schematic representation by Chris Eilbeck



A soliton in a water tank (Bourgogne, France)

Some history - continued

- **Russell to mathematicians:** “... it was not to be supposed that after its existence had been discovered and phenomena determined, endeavors would not be made to reconcile it with existing theory, or to show how it ought to have been predicted from the known equations of fluid motion. In other words, it now remained for the mathematician to predict the discovery after it had happened..."

- Leading British fluid dynamics researchers **doubted** the importance of Russell's solitary wave

- **Sir George Airy (English) 1841**:
  Russel’s wave was **linear**

Sir George Airy

Some history - continued

1847 – Sir George Stokes (Irish): Stokes set up the correct nonlinear water wave equations and found a traveling periodic wave to these equations, where **the speed depends on amplitude.**

- Stokes made many other critical contributions to fluid dynamics:
  "Navier-Stokes equations"

Claude-Louis Navier (French)

Sir George Stokes

Some history - continued

1871-1877 - Joseph Boussinesq (French): new nonlinear equations and solitary wave solution for shallow water waves

1895 - Diederick Korteweg and Gustav de Vries (both Dutch): simplified shallow water wave equation ("**KdV equation**"); nonlinear periodic solution: **"cnoidal" wave**; special case being the solitary wave solution. **Russell's work was (finally) confirmed!**

Joseph Boussinesq

Diederick Korteweg

Gustav de Vries

From water waves to nonlinear lattices: Fermi-Pasta-Ulam (FPU)

In 1955, Fermi, Pasta, Ulam (FPU) and Tsingou undertook a numerical study (Los Alamos) of a one-dimensional anharmonic (nonlinear) lattice.

They thought that due to the nonlinear coupling, any smooth initial state would eventually lead to an **equipartition of energy**, i.e., a smooth state would eventually lead to a state whose harmonics would have equal energies.

In fact, they did not see this in their calculations. What they found is that the solution nearly recurred and the energy remained in the lower modes!

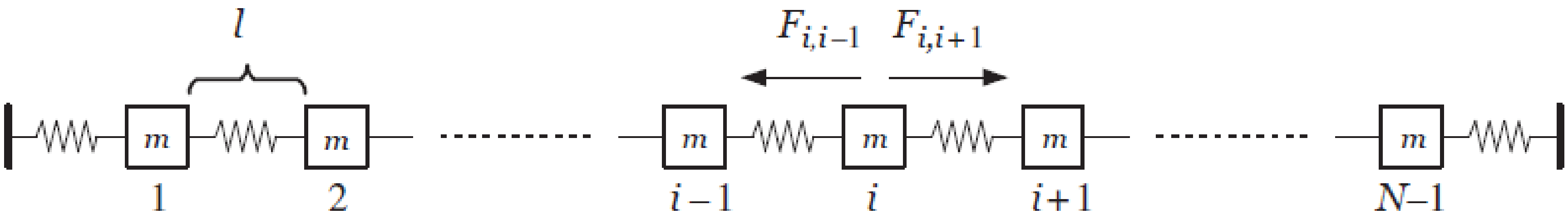
Enrico Fermi

John Pasta

Stanislaw Ulam

Mary Tsingou

The FPU model



The FPU model consists of a nonlinear spring–mass system with the force law: $F(\Delta)=-k(\Delta+\alpha\Delta^2)$; here, Δ is the displacement between the masses, $k > 0$ is constant, and α is the nonlinear coefficient. Using Newton's 2nd law, one obtains the following equation governing the longitudinal displacements:

$$m\ddot{y}_i = k\left[(y_{i+1}-y_i)+\alpha(y_{i+1}-y_i)^2\right]-k\left[(y_i-y_{i-1})+\alpha(y_i-y_{i-1})^2\right],$$

$$m\ddot{y}_i = k(y_{i+1}-2y_i+y_{i-1})+k\alpha\left[(y_{i+1}-y_i)^2-(y_i-y_{i-1})^2\right],$$

$$\frac{m}{k}\ddot{y}_i = \hat{\delta}^2 y_i + \alpha\left[(y_{i+1}-y_i)^2-(y_i-y_{i-1})^2\right], \quad \hat{\delta}^2 y_i \equiv (y_{i+1}-2y_i+y_{i-1})$$

$$N = 65 \qquad y_i(t=0) = \sin\left(\frac{i\pi}{N}\right), \qquad \dot{y}_i(t=0)=0, \qquad i = 1,2,\ldots,N-1,$$

The work of Zabusky and Kruskal

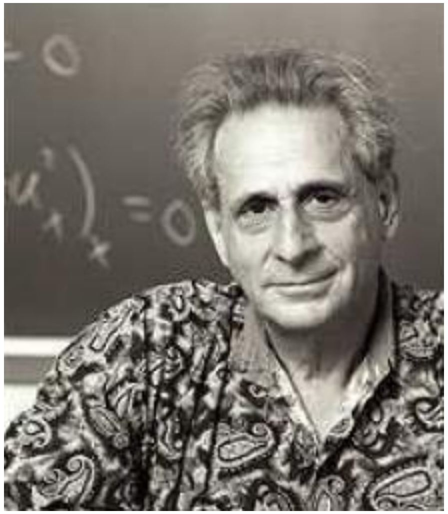
In **1965, Zabusky** and **Kruskal** studied the **continuum limit** corresponding to the FPU model: They considered y as approximated by a continuous function of the position and time and expanded y in a Taylor series,

$$y_{i\pm1} = y((i\pm1)l) = y \pm l y_z + \frac{l^2}{2} y_{zz} \pm \frac{l^3}{3!} y_{zzz} + \frac{l^4}{4!} y_{zzzz} + \cdots \quad z = il$$

Normalizations: $h = l/L,\ x = z/L,\ L = Nl,\ t = \tau/(h\omega),\ \omega = \sqrt{k/m},$

$$h^2 y_{\tau\tau} = h^2 y_{xx} + \frac{h^4}{12} y_{xxxx} + \alpha\left[\left(h y_x + \frac{h^2}{2} y_{xx} + \ldots\right)^2 - \left(h y_x - \frac{h^2}{2} y_{xx} + \ldots\right)^2\right].$$

Norman Zabusky



Martin Kruskal

Zabusky and Kruskal - continued

- Continuous limit, to leading order:

$$y_{\tau\tau} = y_{xx} + \frac{h^2}{12} y_{xxxx} + \varepsilon y_x y_{xx} + \cdots \quad \overbrace{h = l/L,}^{\text{dispersion}} \; \overbrace{\varepsilon = 2\alpha h}^{\text{nonlinearity}}$$

➢ Derived by **Boussinesq (1871-1872)** for **shallow-water waves!**

- **Four interesting cases:**

➢ **No** dispersion, **no** nonlinearity: $h^2 \ll 1 \quad |\varepsilon| \ll 1 \quad \boxed{y_{\tau\tau} = y_{xx}.}$

➢ Dispersion, **no** nonlinearity: $h^2/12 \gg |\varepsilon| \quad \boxed{y_{\tau\tau} = y_{xx} + \frac{h^2}{12} y_{xxxx}.}$

➢ **No** dispersion, nonlinearity: $h^2/12 \ll |\varepsilon|, \quad \boxed{y_{\tau\tau} = y_{xx} + \varepsilon y_x y_{xx}}$

➢ Dispersion, nonlinearity: balance $h^2/12 \approx |\varepsilon| \ll 1 \Rightarrow$ **SOLITONS!**

From Boussinesq to KdV

Boussinesq equation

$$y_{\tau\tau} = y_{xx} + \frac{h^2}{12} y_{xxxx} + \varepsilon y_x y_{xx}$$

Solution of the form

$$y \sim \Phi(X, T; \varepsilon), \;\; X = x - \tau, \;\; T = \frac{\varepsilon\tau}{2}$$

$$\left[\frac{\partial^2\Phi}{\partial X^2} - \varepsilon \frac{\partial\Phi}{\partial X \partial T} + \frac{\varepsilon^2}{4}\frac{\partial^2\Phi}{\partial T^2}\right] = \frac{\partial^2\Phi}{\partial X^2} + \frac{h^2}{12}\frac{\partial^4\Phi}{\partial X^4} + \varepsilon \frac{\partial\Phi}{\partial X}\frac{\partial^2\Phi}{\partial X^2}.$$

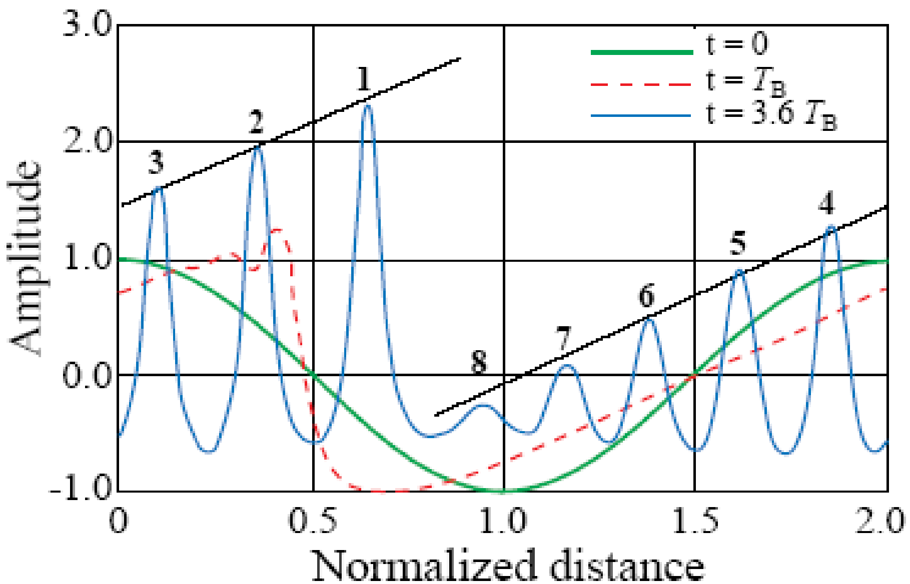
Let $u = \partial\Phi/\partial X$, drop $O(\varepsilon^2)$ terms, and define: $\delta^2 = h^2/12\varepsilon$

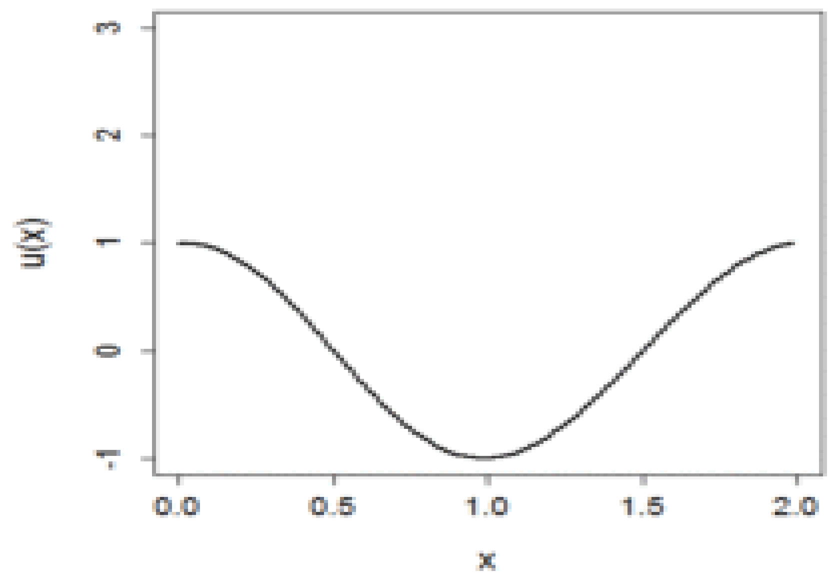
$$u_T + u u_X + \delta^2 u_{XXX} = 0$$ KdV equation!

A note: Before early 1960's, KdV was not of wide interest; waves were chiefly studied by means of **linear, 2nd-order** equations, while KdV is **nonlinear** and **3d-order** equation!

KdV solitons and the FPU "paradox"

- Zabusky & Kruskal: $u_T + uu_X + \delta^2 u_{XXX} = 0 \quad u(X,0) = \cos(\pi X)$





- When $\delta^2 << 1$, a sharp gradient appears at a finite time, $t = t_B$, together with "wiggles".

- When $t >> t_B$, the solution develops many oscillations that eventually separate into a **train of solitary-type waves**. Each solitary wave is localized in space.

- Subsequently, under further propagation, the solitary waves interact and the **solution eventually returns to a state that is similar to the initial conditions**, one which resembles the **recurrence phenomenon** observed by **FPU** in their computations!

The seminal paper

VOLUME 15, NUMBER 6 PHYSICAL REVIEW LETTERS 9 AUGUST 1965

INTERACTION OF "SOLITONS" IN A COLLISIONLESS PLASMA AND THE RECURRENCE OF INITIAL STATES

N. J. Zabusky

Bell Telephone Laboratories, Whippany, New Jersey

and

M. D. Kruskal

Princeton University Plasma Physics Laboratory, Princeton, New Jersey

(Received 3 May 1965)

A note: the term **"soliton"**, originating from **"solit-ary"** and **"on"** (usually used for particles), was first introduced in the paper of Zabusky and Kruskal

Solitons – anything else?

Solitons in nonlinear optics: the nonlinear Schrödinger (NLS) equation

- **Temporal solitons in optical fibers** $n \equiv \frac{kc}{\omega} = n_0(\omega) + n_2|E|^2$

$$i\left(\frac{\partial E}{\partial z} + k'\frac{\partial E}{\partial t}\right) - \frac{k''}{2}\frac{\partial^2 E}{\partial t^2} + \frac{\omega_0 n_2}{c}|E|^2 E = 0$$

Dispersion Nonlinearity

- **Spatial solitons in bulk media and waveguides** $n = n_0 + n_{NL}\left(|E|^2\right)$

$$2ik_0 n_0 \frac{\partial E}{\partial z} + \left(\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2}\right) + 2n_0 k_0^2 n_{NL}\left(|E|^2\right)E = 0$$

Diffraction Nonlinearity

Solitons of the cubic NLS model

- **NLS: Completely integrable – exact soliton solutions**

❑ **Focusing** ($s\sigma$ = -1) and **defocusing** ($s\sigma$ = +1) NLS

$$i\psi_t - \frac{s}{2}\psi_{xx} + \sigma|\psi|^2\psi = 0, \qquad \sigma, s = \pm 1$$

- $s\sigma$ = -1 : **Bright Solitons**

$$\psi(x,t) = \eta\,\mathrm{sech}\left[\eta(x-kt)\right]\exp\left[i(kx-\omega t)\right] \quad \omega = \left(k^2-\eta^2\right)/2$$

- $s\sigma$ = +1 : **Dark Solitons**

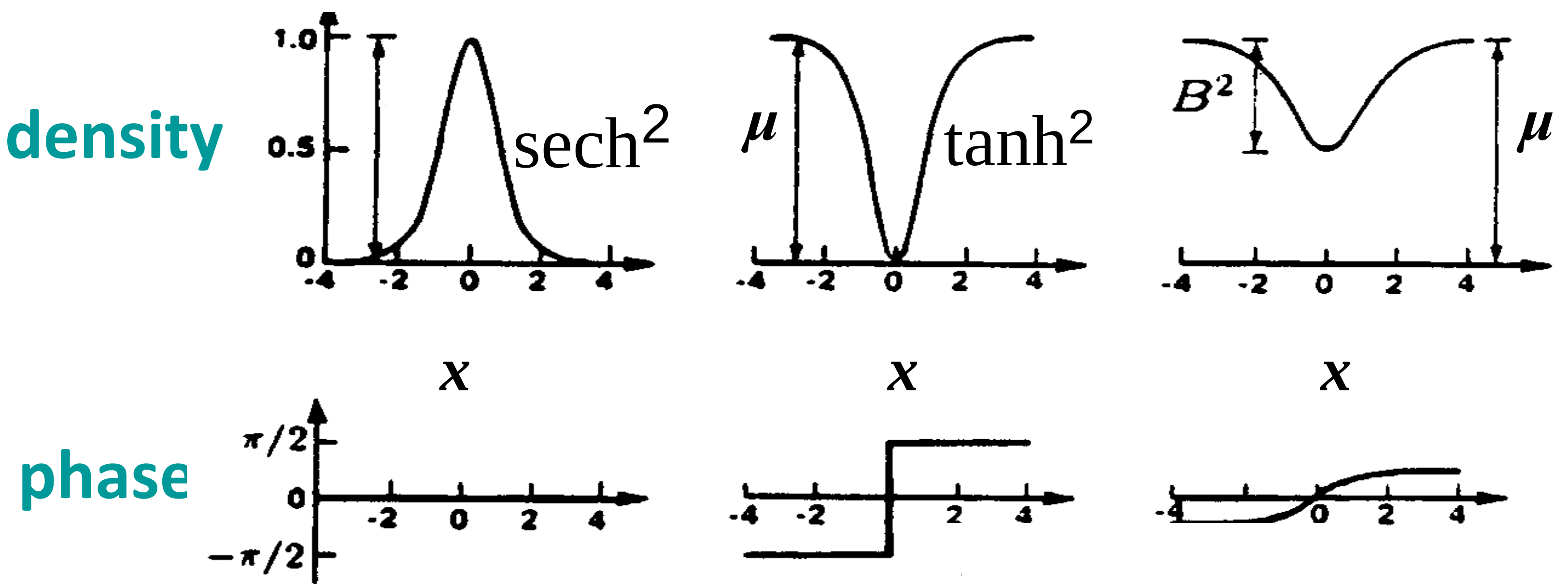
$$\psi(x,t) = \sqrt{\mu}\left(B\tanh\xi + iA\right)\exp\left(-i\mu t\right)$$

$$\xi = \psi_0 B\left(x - \sqrt{\mu}At\right) \quad A^2 + B^2 = 1$$

Types of solitons of the NLS equation

Bright solitons   Dark solitons

black   **gray**

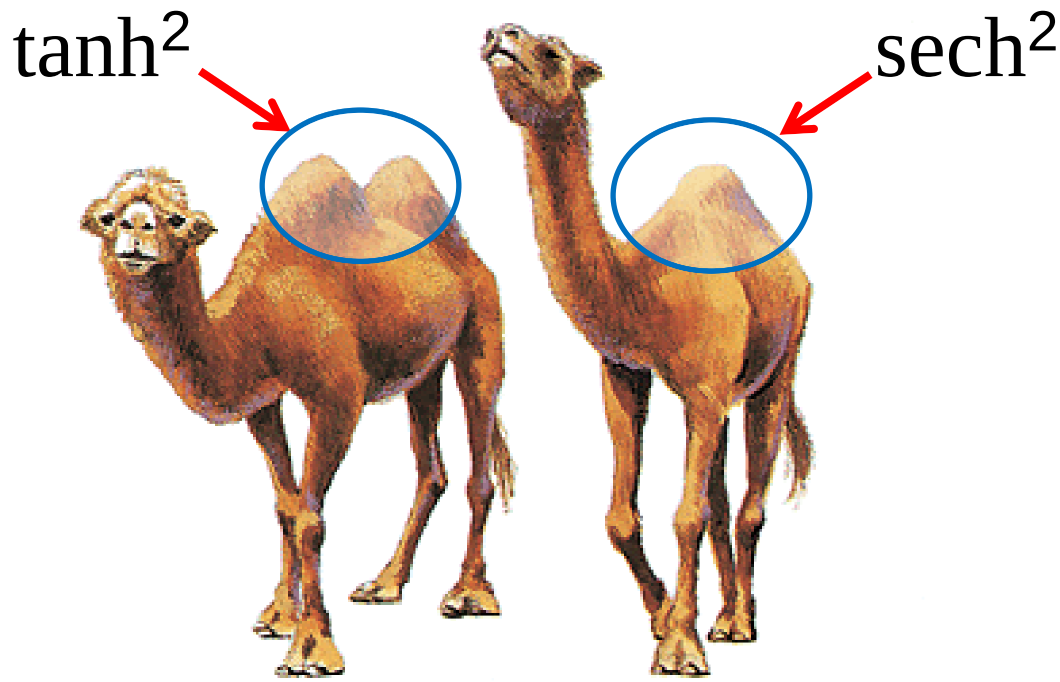


Bright solitons: Nontopological solitons

Dark solitons: Topological solitons (phase kinks)

Bright and Dark Solitons

Another interpretation:

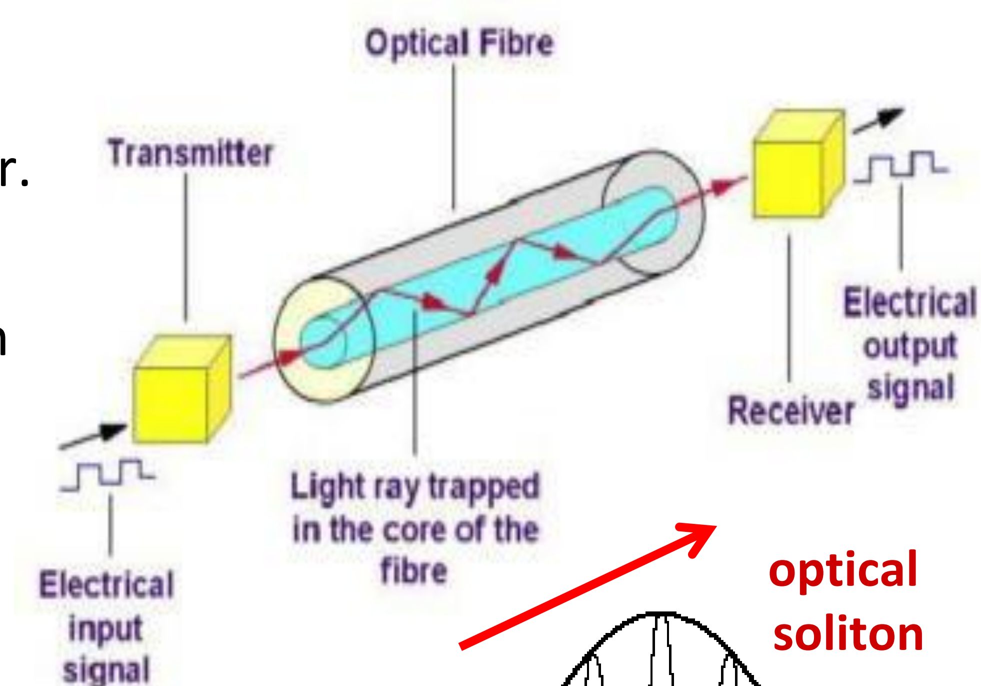


Different animals in the jungle of nonlinear waves!

Temporal solitons in optical fibers

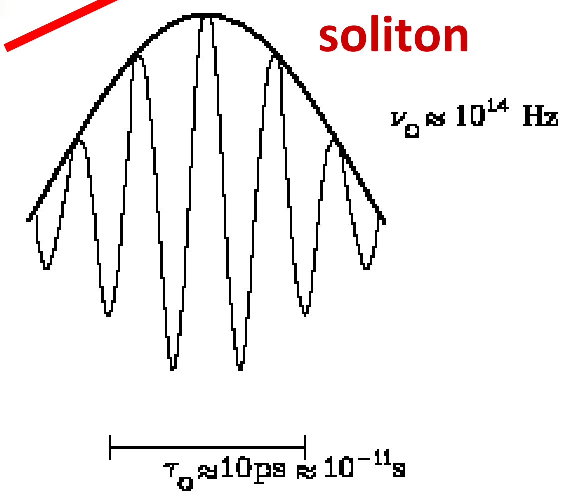
Optical fibers

- Consist of: special form of silica glass.
- Size: no thicker than a human hair.
- Cost: 20 euros per kilometer.
- Its job: is to guide the light (which carries the information) with minimum attenuation or loss



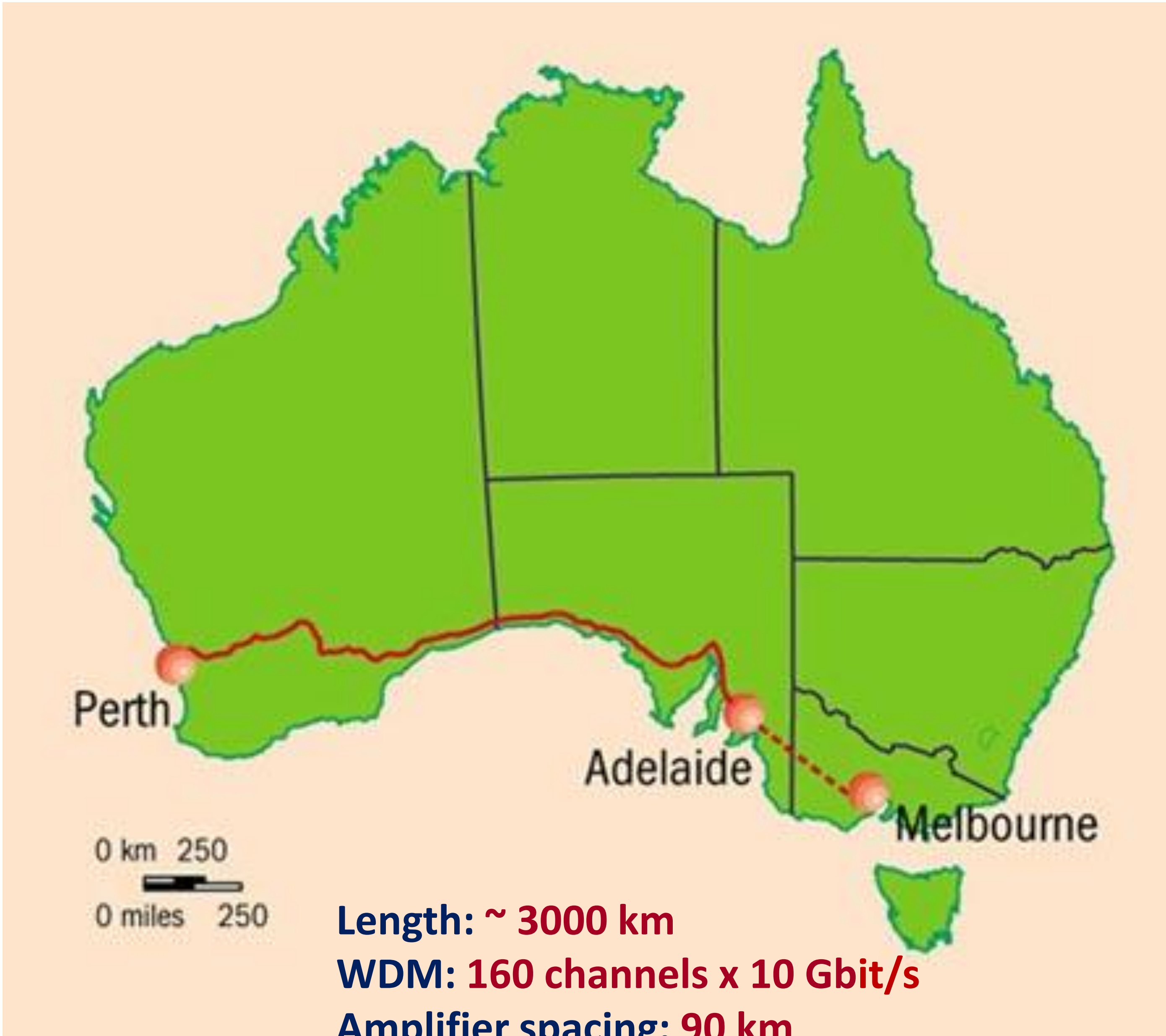
Optical fibers are **dispersive nonlinear media** $\Rightarrow$ **solitons!**

optical soliton



$$i\left(\frac{\partial E}{\partial z}+k'\frac{\partial E}{\partial t}\right)-\frac{k''}{2}\frac{\partial^2 E}{\partial t^2}+\frac{\omega_0 n_2}{c}|E|^2 E=0$$

A long optical fiber link with solitons!

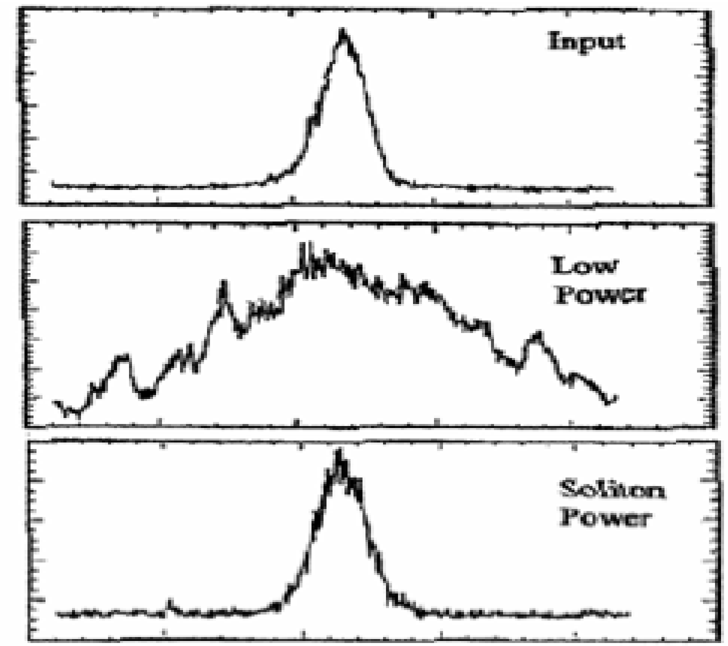


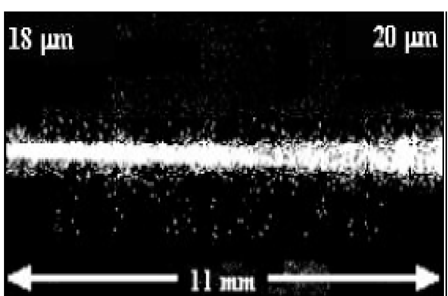
Length: ~ 3000 km
WDM: 160 channels x 10 Gbit/s
Amplifier spacing: 90 km

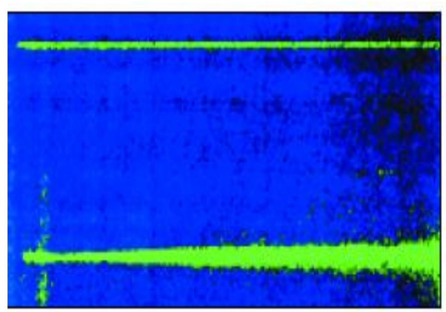
Optical spatial solitons

Bright solitons

Non-diffracting optical beams



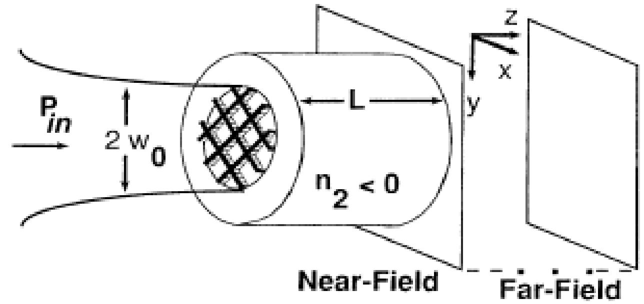


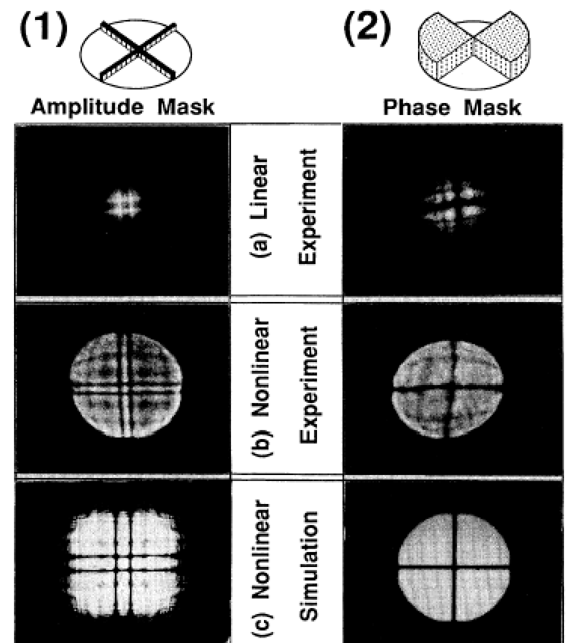


Self-focusing: $n = n_0 + n_2 I$

Dark solitons

Non-diffracting "shadows"

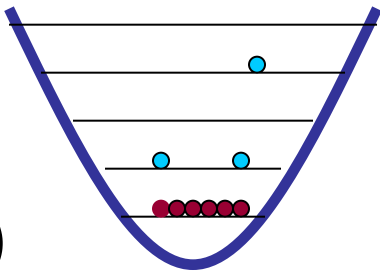




Self-defocusing: $n = n_0 - n_2 I$

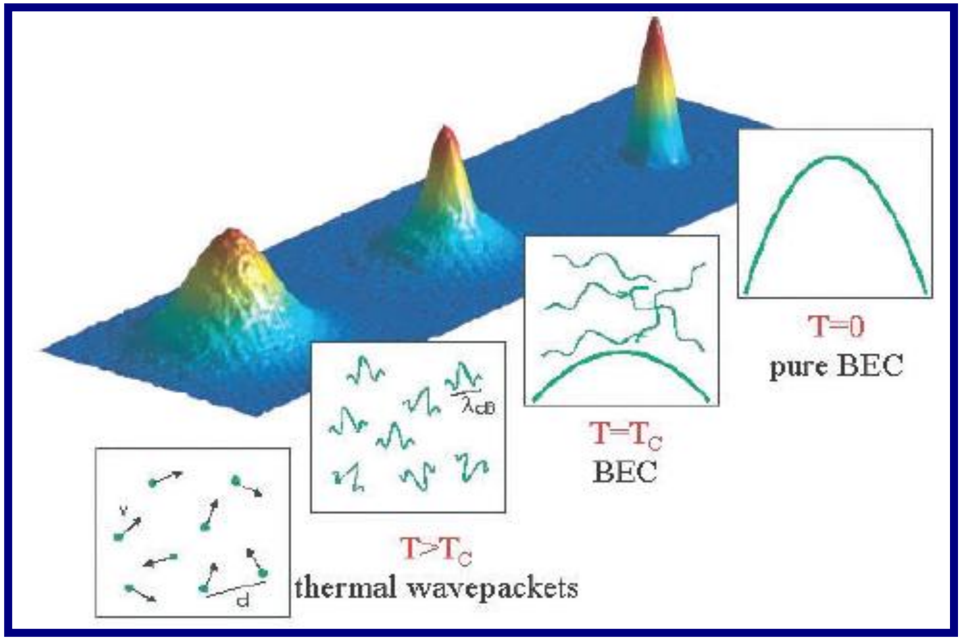
The quantum world: Bose-Einstein condensates

- **Bose-Einstein condensate (BEC)**: *A state of matter in which a "macroscopic" number of particles* (10^3 - 10^4) *obeying the Bose-Einstein statistics (bosons) share the same quantum state of lowest-energy (for* $T << T_c \approx$ 100nK)



- **Original theoretical prediction: Bose - Einstein** (1925)

- **Experimental observation: Cornell-Wieman-Ketterle-Hulet** (1995) in dilute alkali gases (^{87}Rb, ^{23}Na and ^{7}Li) - ***Nobel prize in Physics* (2001)**



Bose-Einstein condensate is:

- **The coldest matter in the Universe (0.5 < T < 100 nK)**
- **A "macroscopic" quantum system (R ~ 50 µm, L ~ 300 µm)**

BECs in the mean-field picture ($T \to 0$)

- **BEC wavefunction** $\Psi(\vec{r},t)$: product of N single-particle wavefunctions
- **Gross-Pitaevskii equation (GPE) :** ***Nonlinear Schrödinger equation***

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r}) + g|\Psi|^2 \right]\Psi$$

Kinetic energy term

External trapping potential

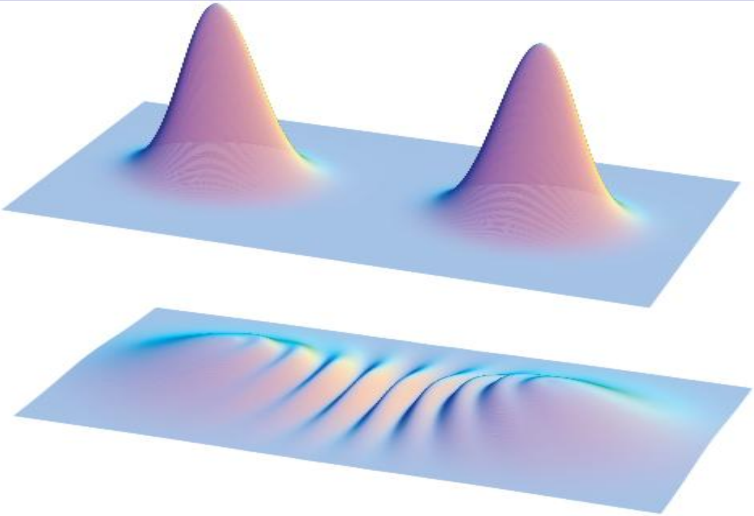
$$V(\vec{r}) = \frac{1}{2} m\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right)$$

Interatomic interaction term

$g > 0$: **repulsive BEC**

$g < 0$: **attractive BEC**

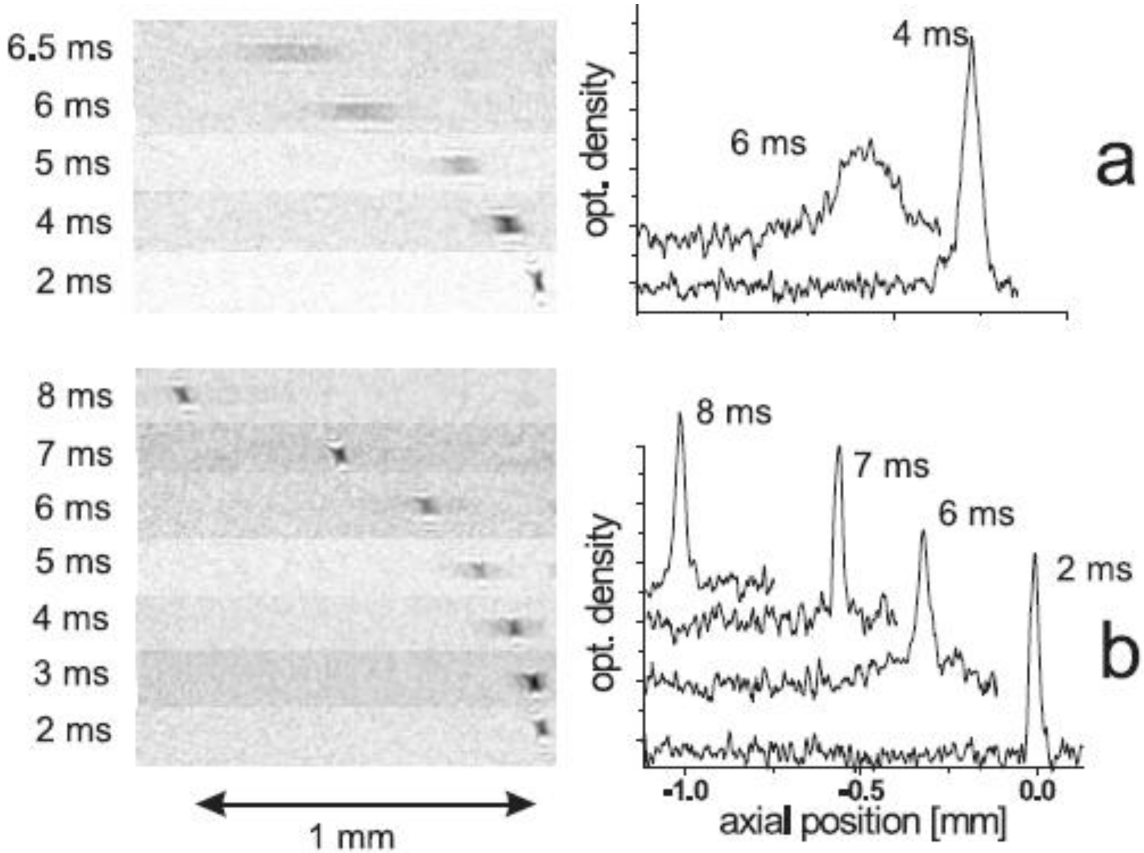
$$g = \frac{4\pi\hbar^2 \alpha}{m}$$

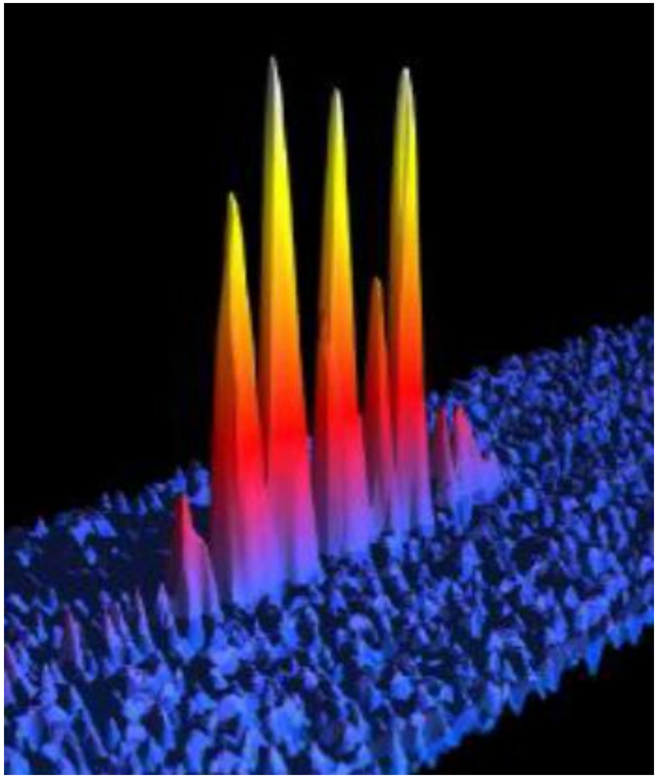


BEC is a *coherent matter-wave* similar to *laser*:

- ✓ **Atoms in a BEC behave the same way similarly to photons in a laser beam**
- ✓ **Independent BECs interfere like laser beams (MIT group, Science 1997)**

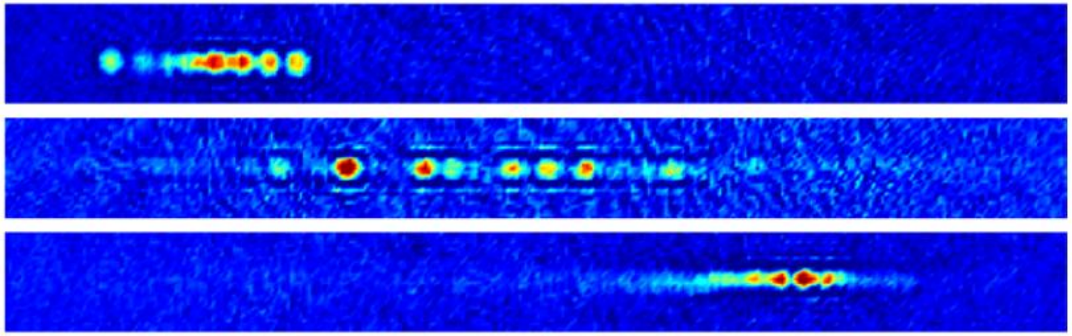
Bright solitons in attractive BECs



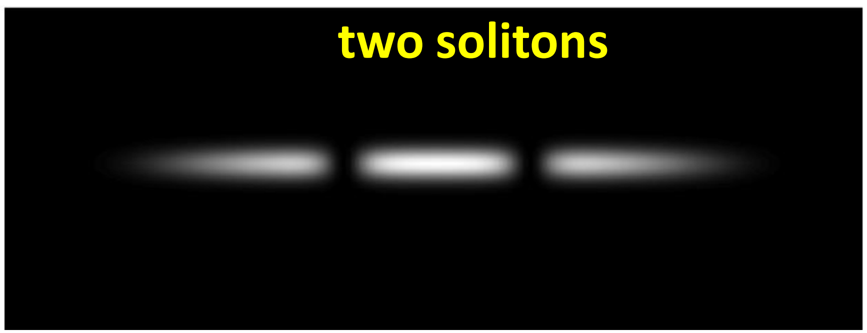


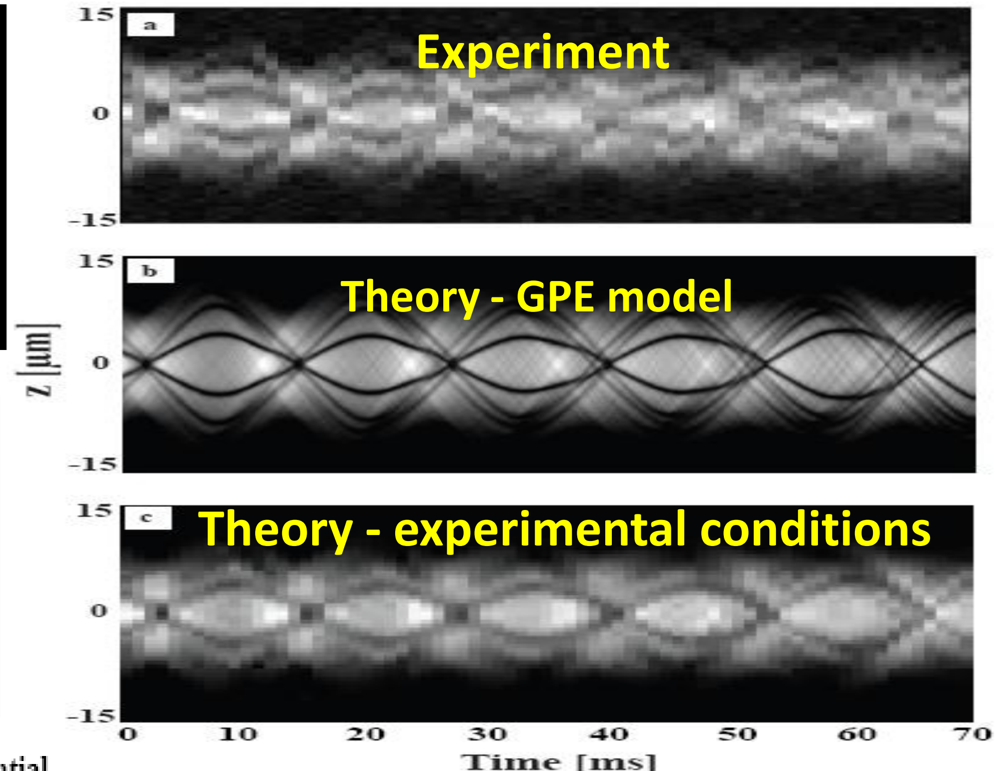
Rice group, Nature 2002

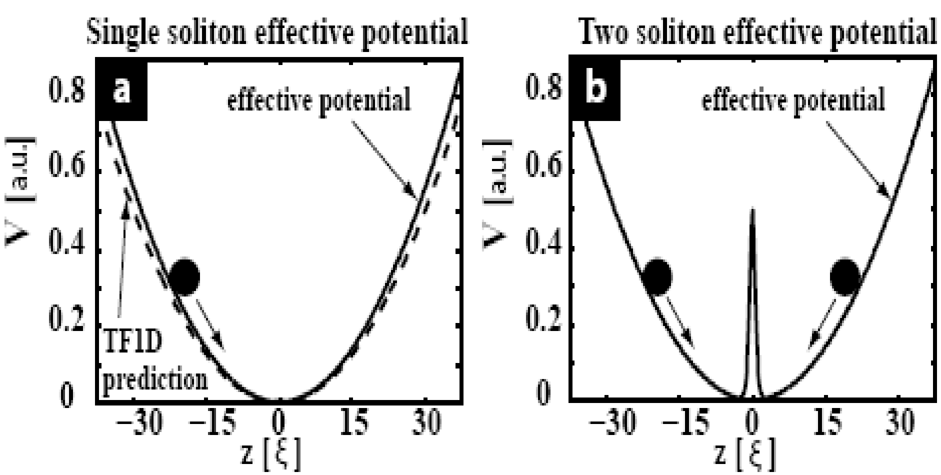
ENS group, Science 2002



Dark solitons in repulsive BECs







Dark solitons in BECs have been made to **live long enough** (up to 1-2 seconds!) for their dynamical properties to be observed. They may serve as a **sensitive probe of the properties of quantum systems!**

Solitons: can we hear them?
The sound of solitons

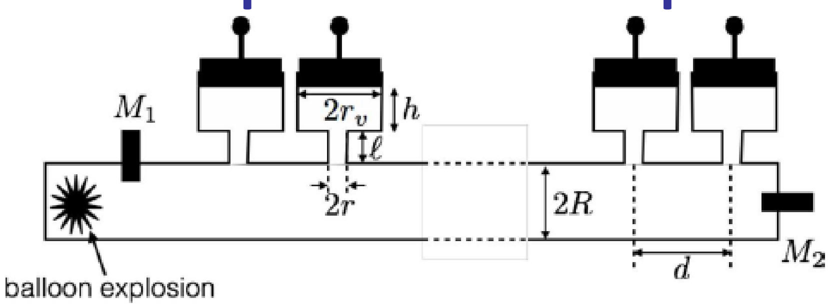
In acoustics:

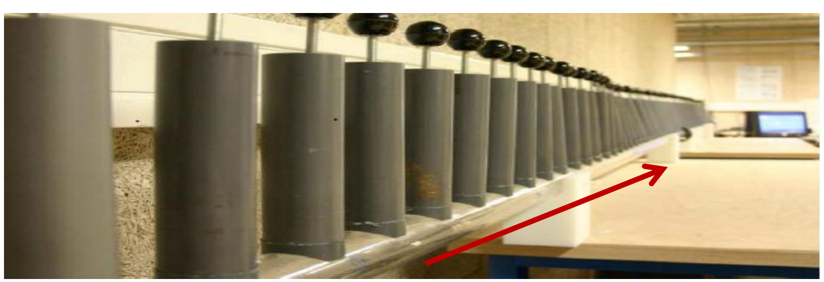
❖ **Typical nonlinear waves: shock waves**

✓ Shock waves emerge in air-filled tubes (as, e.g., in train tunnels – TGV net in France)

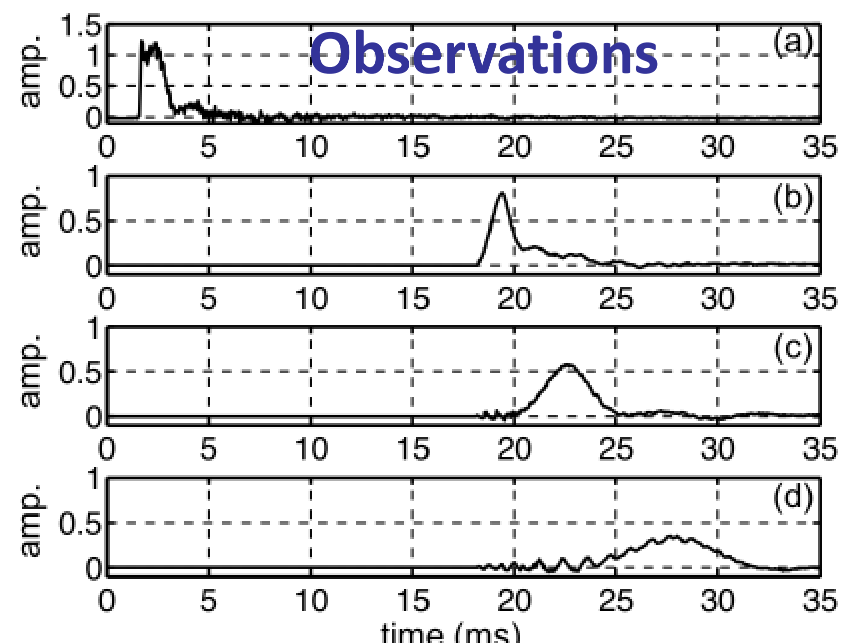
➢ Can we eliminate formation of shocks by adding "**inclusions**" in the tube?

Experimental setup





Observations



Modeling and results

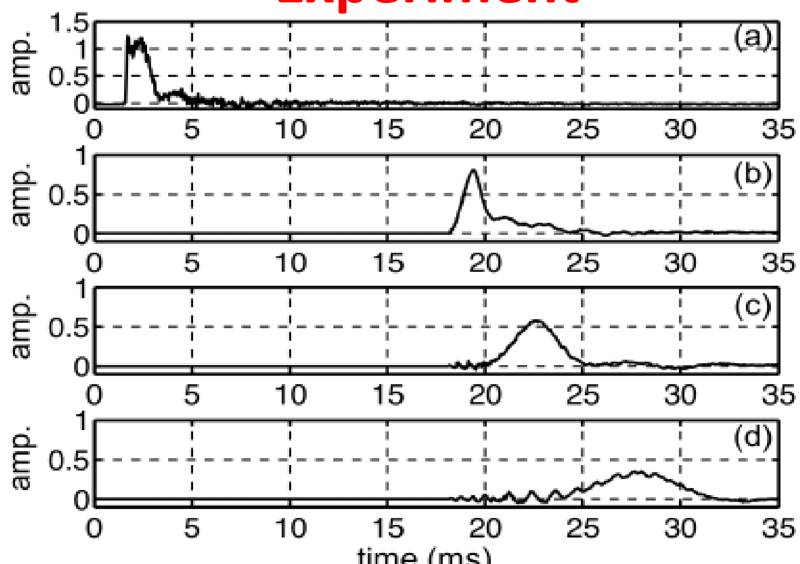
$$P_{\tau\tau} - P_{\chi\chi} - \Omega^2(P_{\chi\chi\tau\tau} - \alpha P_{\tau\tau\tau\tau}) - \epsilon\alpha(P^2)_{\tau\tau} = 0$$ **Boussinesq (!)**

$$P(\chi,\tau) = \Phi(\xi) \quad \xi = \delta(\tau - \chi/v) \quad \Longrightarrow \quad A\Phi'' + B\Phi - \epsilon\alpha\Phi^2 = 0$$

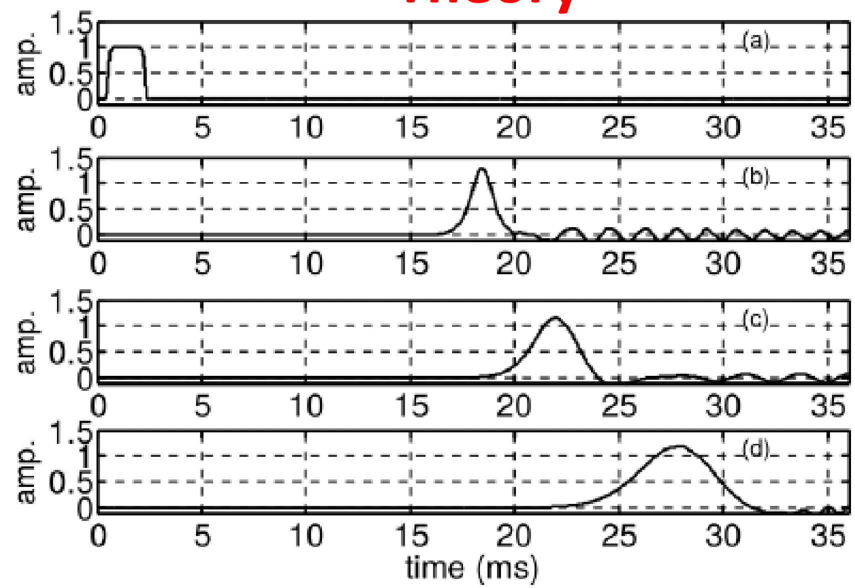
$$P(\chi,\tau) = \left(\frac{\Omega^2}{\epsilon}\right)\left(\frac{6\kappa\delta^2}{1+4\delta^2\Omega^2}\right)\mathrm{sech}^2\left[\delta\left(\tau - \frac{\chi}{v}\right)\right]$$

$$\delta = \sqrt{\epsilon} \ll 1 \quad \Longrightarrow \quad P_{1X} - \frac{\Omega^2}{2}(1-\alpha)P_{1TTT} - \alpha P_1 P_{1T} = 0$$ **KdV (!)**

Experiment



Theory

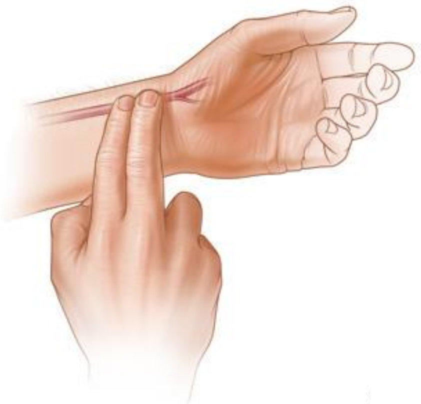


Solitons inside us (!)

Blood-pressure waves

- Heart sends a pressure wave in the arteries, inducing a local expansion of the vessel.
- The deformation in the elastic vessel tube is felt as the pulse

❖ Question: why is it possible to feel the blood pulse at the wrist?



➢ Answer: because ***blood pressure waves*** are described by **KdV solitons**, which are quite robust, and propagate without distortion in arteries!

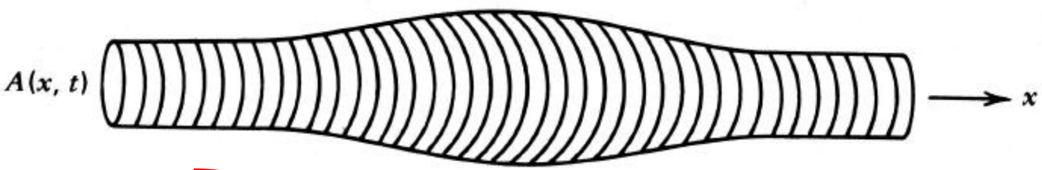
Modeling and results

- <u>Assumptions:</u>

❖ blood is an incompressible and inviscid fluid

❖ the artery is an infinitely long circular elastic cylinder

❖ the flow is 1D (localized pressure increase causes radially symmetric expansion)



- <u>Analysis:</u>

$$A_t + (Av)_x = 0$$

$$v_t + vv_x = -p_x$$

$$p = (A-1) + A_{tt}$$

velocity

pressure

cross section of the artery

KdV equation!

$$\boxed{p_\tau + \frac{3}{2}pp_\xi + \frac{1}{2}p_{\xi\xi\xi} = 0}$$

Pressure pulse: $p(x,t) = \dfrac{chE}{a}\,\mathrm{sech}^2\left[\sqrt{\dfrac{k^2c\rho_0}{2ah\rho_R}}\,x - \sqrt{\dfrac{k^2cE}{4a^2\rho_R}}(1+k^2c)\,t\right]$

Smoking $\Rightarrow$ arteries become more rigid $\Rightarrow$ E increases $\Rightarrow$ blood pressure and speed increases

Fat accumulation $\Rightarrow$ thickness of artery wall h increases $\Rightarrow$ pulse width & blood pressure increase

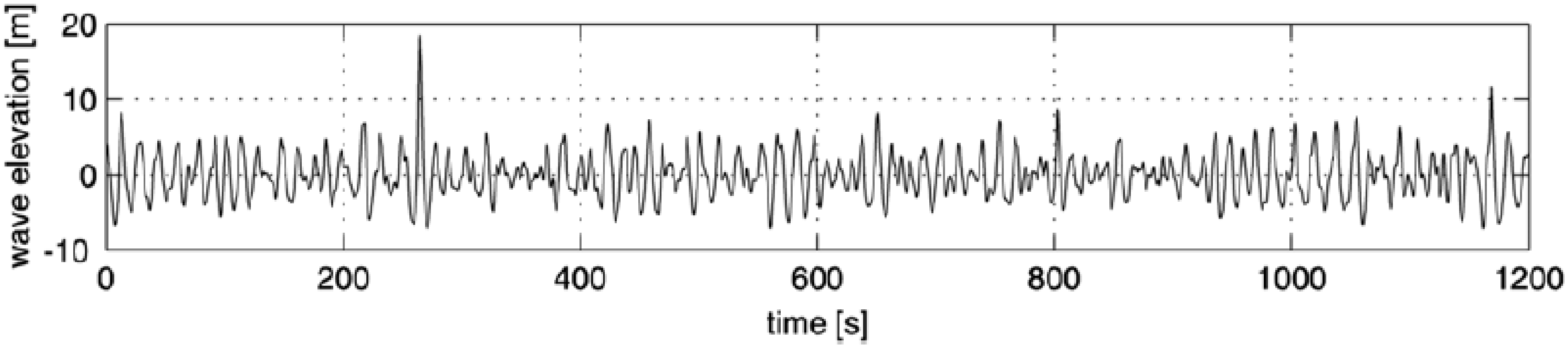
Solitons localized in <u>both</u> space & time:
Rogue / freak / extreme waves

Giant waves that appear from nowhere and disappear without a trace!

Reports on extreme wave events

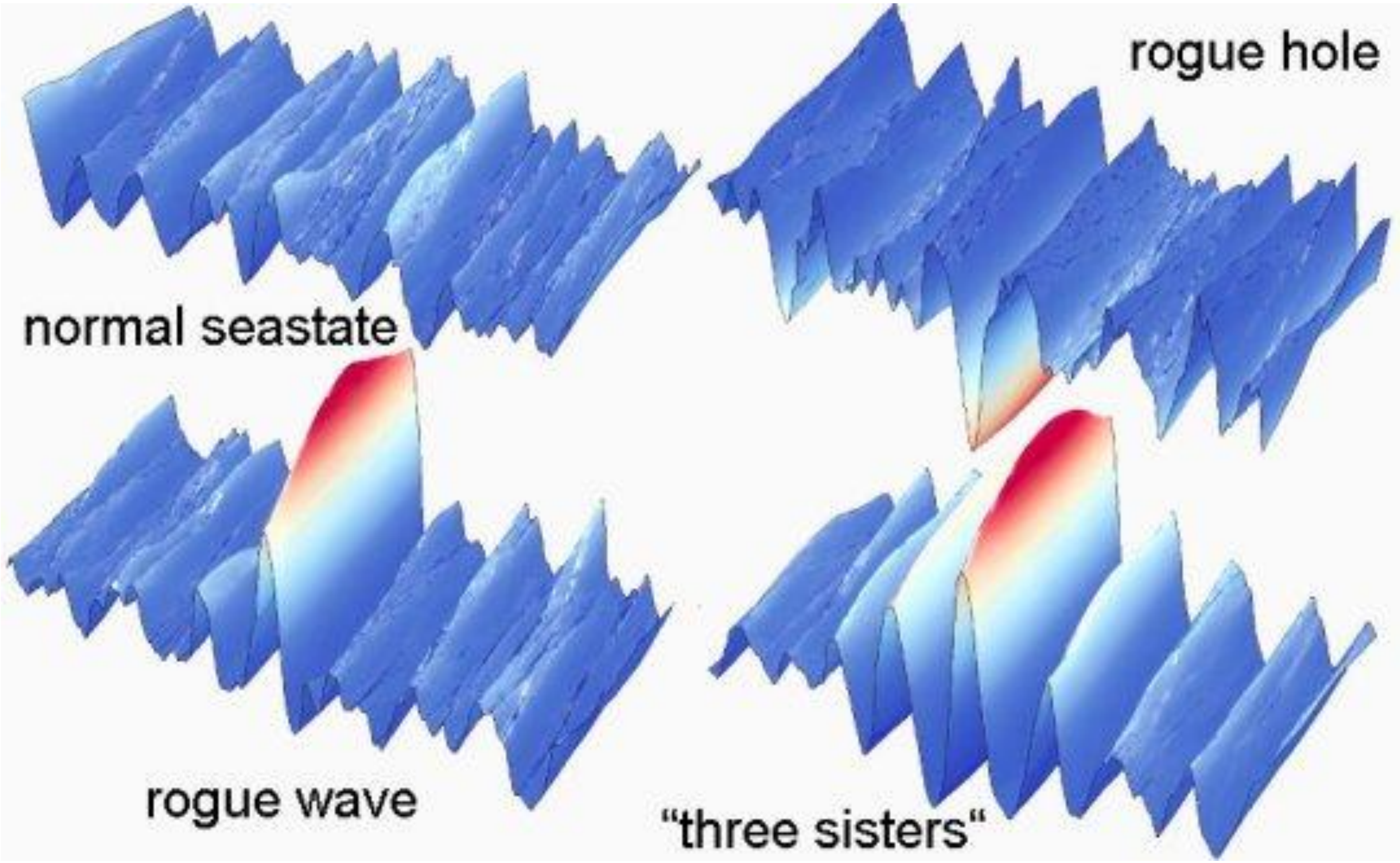
- According to seafarer and fishermen stories <u>(here from a pub in Ireland)</u>, rogue waves like solid walls of water, higher than 30 meters, or holes in the sea, are more or less common phenomena in deep ocean waters.

  - ✘ Anything more reliable?
  - ✓ Data from satellites and oil platforms!

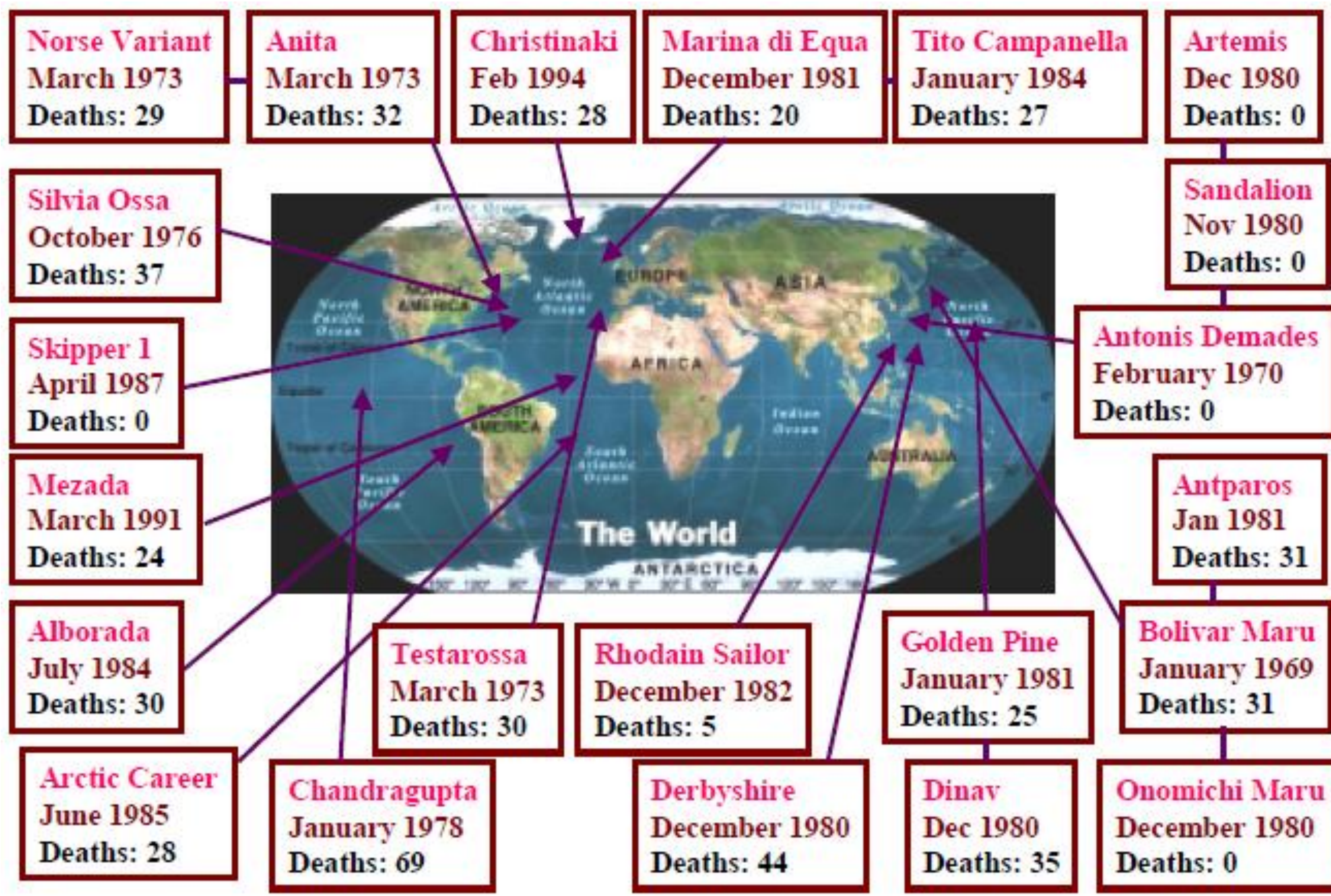


- The "New Year Wave", 26m height in the North Sea
- Recorded at Draupner platform, Norway, January 1st, 1995

Usual forms of rogue waves in the sea



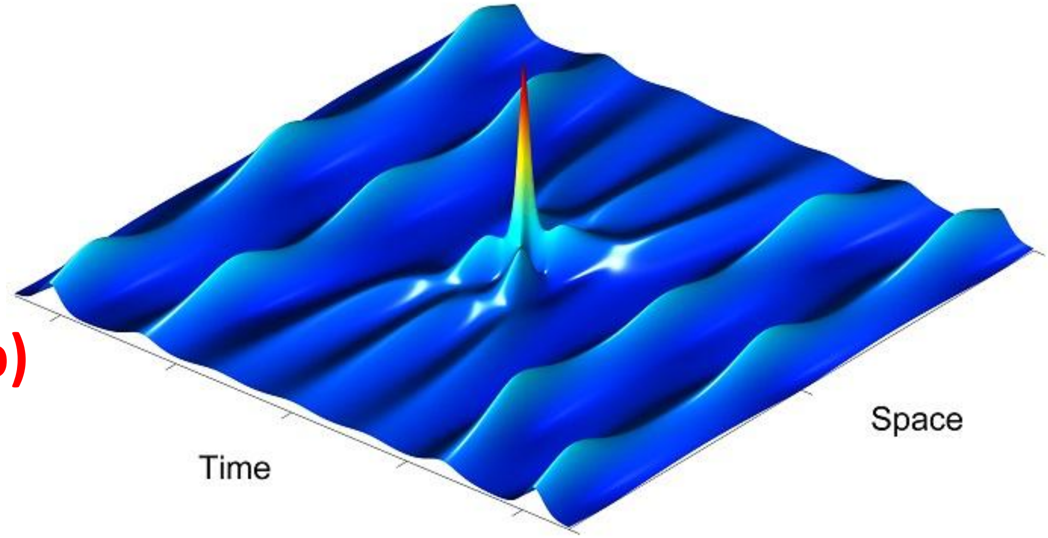
Statistics of collisions with rogue waves



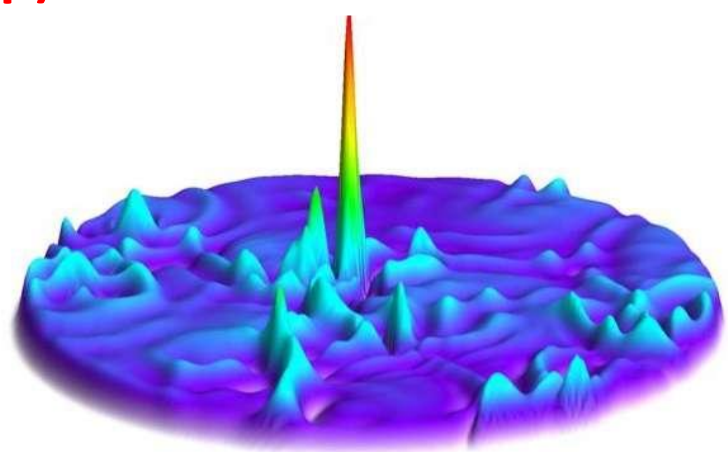
Other manifestations of rogue waves

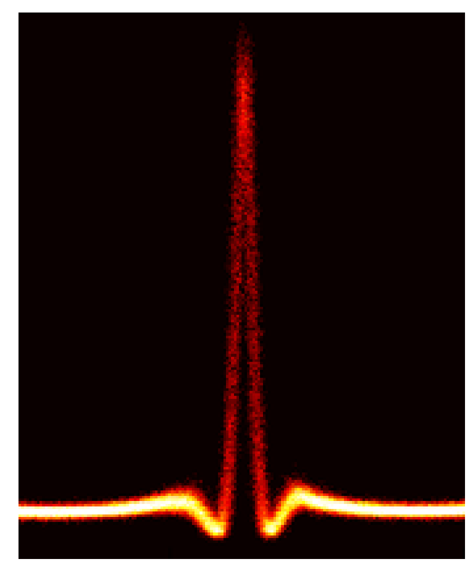
Rogue waves also have been found in:

- ✓ Nonlinear optics (exp)
- ✓ Mode-locked lasers (exp)
- ✓ Superfluid helium (exp)
- ✓ Hydrodynamics (exp)
- ✓ Faraday surface ripples (exp)
- ✓ Parametrically driven capillary waves (exp)
- ✓ Plasmas (exp)



- ✓ BECs (th)
- ✓ Econophysics (th)
- ✓ …





Genuinely interdisciplinary research topic!

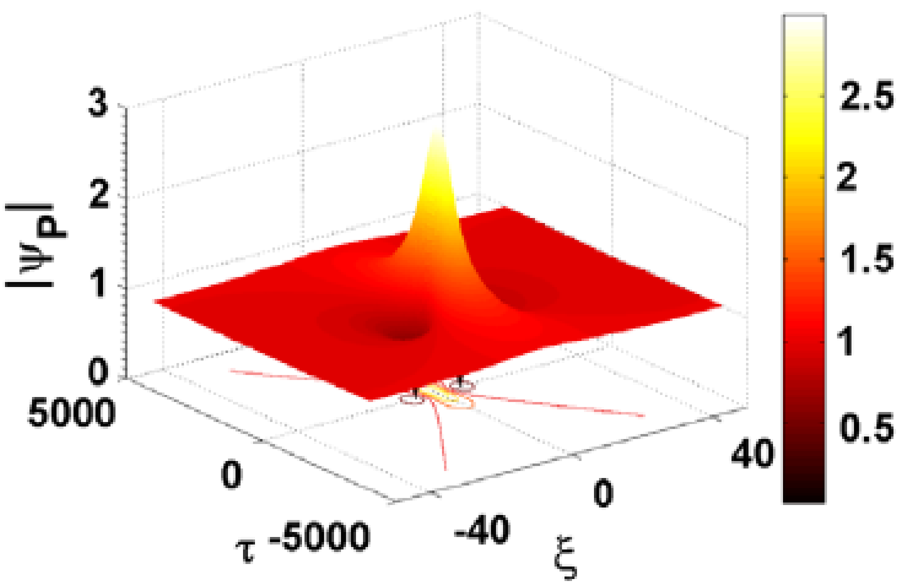
The Peregrine soliton of the NLS

NLS equation
Deep water waves
Nonlinear optics

$$\mathrm{i}\frac{\partial \psi}{\partial \tau} + P\frac{\partial^2 \psi}{\partial \xi^2} + Q|\psi|^2 = 0$$

Peregrine soliton: localized both in space AND in time

$$\psi(\xi,\tau) = \left[1 - \frac{4(1+\mathrm{i}2Q\tau)}{1+2Q\xi^2/P+4Q^2\tau^2}\right]\exp(\mathrm{i}Q\tau)$$



Conclusions

- **Solitons are extraordinary waves:**

  they propagate **undistorted** in **nonlinear dispersive media** and feature **particle-like properties** (they undergo **elastic collisions**)

- **Historical discussion:**

  from **Euler** to **J. Scott Russell** and his “great wave of translation”, and from **water waves** to **nonlinear lattices** and the **FPU** paradox, and **beyond**

- **Solitons are around us - and even inside us (!) :**

  - there are many cases/situations where solitons **can be observed**;
  - they **appear ubiquitously in all branches of physics** and in **all scales**
  - they can be used to understand the **emergence of coherent robust structures in nature**
  - they can be used in **applications** (e.g., optical fiber communications)

A lot of things have been understood…
and many more remain to be done…

Thank you!