Non-extensive entropies

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0. Outline

1. Statistical basis of entropy: Boltzmann, Gibbs, etc.
2. Dynamical basis of entropy: Boltzmann vs Landau/Khitchin.
3. Axioms for S_{BGS}: C. Shannon and A. I. Khintchin.
4. Some Formal Properties of S_{BGS}.
5. Critique of the BGS entropy.
6. An "alternative": a non-additive entropy.
7. Some other entropic functionals.
8. Some Formal Properties of S_q.
9. Axioms: R.J.V. Santos and S. Abe.
10. Dynamical and Physical Foundations of S_q.
11. Some Weaknesses (limitations?) of S_q.
12. Some References ...

1. Statistical basis of entropy: Boltzmann, Gibbs, etc

- What is the microscopic origin of entropy ?
- Boltzmann: S is related to the number of microscopic configurations of a system compatible with its macroscopic conditions/constraints.
- To do the counting use log (motivation: ideal gases)

$$S = k_B \log W$$

W: number of micro-states corresponding to the same macro-state.
(normalised phase space volume of hyper-surface of energy E.)

- Contribution: Introduction of probabilistic reasoning in Physics.
- Gibbs: The entropy of an "ensemble" (set of copies of the system (same microscopic dynamics) but different initial conditions) is

$$S = -k_B \sum_{i \in I} p_i \log p_i$$

- For our purposes: Why use log in the definition of entropy ? OK for ideal gases or weakly interacting systems, but why in general?

2. Dynamical basis of entropy: B/E vs L/K.

- Boltzmann/Einstein: the origin of all macroscopic properties can be found in the underlying dynamics.

- How do we actually calculate anything? Molecular Chaos hypothesis, Ergodicity etc.

- Landau/Khintchin: The underlying dynamics is only good to get a probability distribution. Then we can ignore all dynamics. Statistics is what matters: Law of Large numbers, Central Limit Theorem etc.

- Quantification (Khintchin's theorem).

3. Axioms for S_{BGS}: C. Shannon and A. I. Khintchin

- C. Shannon (1948) Uniqueness, A.I. Khintchin (1953) Uniqueness

Postulates (A.I. Khintchin Uniqueness; 1953)

Assume that an entropic form $S(\{p_i\}),\ i \in I$ for a system satisfies

1. $S(\{p_i\})$ is a continuous function of $p_i,\ i \in I$
2. $S(\{p_i = \frac{1}{W}\}) > S(\{p_i = \frac{1}{W'}\}),\ \forall\ i \in I,\ W > W'$
3. $S(p_1, p_2, \ldots, p_n, 0) = S(p_1, p_2, \ldots, p_n)$
4. If A, B are subsystems, their combination has joint probability distribution $\{p_{ij},\ i \in I_A,\ j \in I_B\}$ and marginal distributions $p_i(A) = \sum_{j \in I_B} p_{ij},\ p_j(B) = \sum_{i \in I_A} p_{ij}$, then

$$S(\{p_{ij}\}) \ = \ S(\{p_i\}) + S(\{p_j | p_i\})$$

   with $S(\{p_j|p_i\}) = \sum_{i \in I_A} p_i S(\{p_{ij}/p_i\})$: conditional probability of B given A

Then $$S \ = \ -k \ \sum_{i \in I} p_i \log p_i \qquad \text{(with } k > 0\text{)}$$

4. Some formal properties of S_{BGS}

- $S_{BGS} \geq 0$

- Maximum at equal probabilities.

- Concavity: If $p = tp_1 + (1-t)p_2, \quad t \in [0,1]$ then

$$S_{BGS}[p] \; \geq \; t\, S_{BGS}[p_1] \; + \; (1-t)\, S_{BGS}[p_2]$$

- Additivity: for independent systems, i.e. for $p_{ij} = p_i p_j, i \in I_A, j \in I_B$

$$S_{BGS}[p_{ij}] \; = \; S_{BGS}[p_i] + S_{BGS}[p_j]$$

5. Critique of the BGS entropy

- It has not really been *derived* from first principles.
- Not obvious why it is applicable for long-range interactions, strong spatial or temporal correlations etc.
- Non-equlibrium systems ? Applicable or not ? To which ones? Why?
- For a solution: Keep as much of the thermodynamic formalism as possible, but change the entropic functional. See what happens...
- This is grand "heresy" in many quarters. But ... is it "crazy enough"?
- Questionable practice: Why should even "max-ent" be applicable?
- Entirely possible that one entropic form will not cover "everything". Find its range of applicability, alternatives etc

6. An "alternative": a non-additive entropy

- How does someone come up with such a functional ?
- "Divine inspiration" / "Ingenuity" /Experience/Past work awareness...
- Use simplicity / "Occam's razor" as a guide. It may still be wrong ! (e.g. Standard Model of Particle Physics: neither simple nor elegant)
- Motivation of form via simple ODEs (from C. Tsallis' papers):
  - $\frac{dy}{dx} = 0 \implies y = const.$ (assume $y(0) = 1$ here and below).
  - $\frac{dy}{dx} = 1 \implies y = 1 + x$
  - $\frac{dy}{dx} = y \implies y = e^x$. Its inverse is $\log x$, used in defining S_{BGS}
  - $\frac{dy}{dx} = y^q, \; q \in \mathbb{R}$ gives

$$y = \{1 + (1-q)x\}^{\frac{1}{1-q}}$$

  Its inverse is

$$y = \frac{x^{1-q} - 1}{1-q} \equiv \log_q x, \quad \log_1 x = \log x, \quad x > 0$$

  By analogy with S_{BGS}, use $\log_q$ to define an entropy functional.
- Motivation from Group Theory and Geometry. Growth rate of balls: exponential vs polynomial (but also remember: Grigor'chuk groups)

6. An "alternative": a non-additive entropy

- For discrete set of outcomes, define the functionals

$$S_q[\{p_i\}] \; = \; k_B \; \log_q p_i \; = \; k_B \, \frac{1}{q-1} \left\{ 1 - \sum_{i \in I} p_i^q \right\}$$

- $q \in \mathbb{R}$: non-extensive (or entropic) parameter. It introduces a "bias".
- It was initially studied by Havrda and Charvat (1967), Daróczy (1970), Cressie and Read (1984) and others ... It was "re-discovered" by C. Tsallis (1988). Extensively studied in Thermodynamics, Statistical Mechanics etc since that time. Not without controversy.
- For continuous set of outcomes ρ, use the "straightforward" extension

$$S_q[\rho] \; = \; k_B \, \frac{1}{q-1} \left\{ 1 - \int_\Omega [\rho(x)]^q \; dvol_\Omega \right\}$$

- Some recent controversy about this continuous extension.

7. Some other entropic functionals (indicative) ...

- Renyi entropy:

$$S_q^R[\{p_i\}] = \frac{1}{1-q}\log\left(\sum_{i=1}^{N} p_i^q\right)$$

- Kaniadakis (κ-) entropy:

$$S_\kappa = -\sum_{i=1}^{N} \frac{p_i^{1+\kappa} - p_i^{1-\kappa}}{2\kappa}$$

- Curado entropy:

$$S_q^C[\{p_i\}] = \sum_{i=1}^{N}(1 - e^{-qp_i}) + e^{-q} - 1$$

- Landsberg-Vedral entropy:

$$S_q^{LV} = \frac{1}{1-q}\left(1 - \frac{1}{\sum_{i=1}^{N} p_i^q}\right)$$

8. Some formal properties of S_q

- $S_q \geq 0$
- $\lim_{q\to 1} S_q \;=\; S_{BGS}$
- q introduces a bias.
  - If $q < 1$, then $p_i^q > p_i$, so the rare events ($p_i \sim 0$) are "enhanced".
  - If $q > 1$, then the frequent events ($p_i \sim 1$) are "enhanced".
- Domain: If $q < 0$ only states with $p_i \neq 0$ contribute to S_q.
- Extremal at equal probabilities (max. for $q > 0$, min. for $q < 0$).
- Additivity: If two sets of outcomes are independent ($p_{ij} = p_i p_j$) then

$$\frac{1}{k_B} S_q[\{p_{ij}\}] = \frac{1}{k_B} S_q[\{p_i\}] + \frac{1}{k_B} S_q[\{p_j\}] + \frac{1}{k_B^2}(1-q) S_q[\{p_i\}] S_q[\{p_j\}]$$

  If $q < 1$ then $S_q[\{p_{ij}\}] \geq S_q[\{p_i\}] + S_q[\{p_j\}]$. The opposite for $q > 1$
- S_q is concave for $q > 0$, convex for $q < 0$.
- Observe that $S_{BGS} = -\frac{d}{dx} \sum_{i\in I} p_i^x |_{x=1}$

  Observe that $S_q = -\frac{D}{dx} \sum_{i\in I} p_i^x |_{x=1}$ where $\frac{D}{dx} f(x) = \frac{f(qx)-f(x)}{qx-x}$ is the "Jackson Derivative".

9. Axioms: R.J.V. Santos and S. Abe

- R.J.V. Santos (1997) Uniqueness, S. Abe (2000) Uniqueness.

Postulates (S.Abe Uniqueness; 2000)

Assume that an entropic form $S(\{p_i\}),\ i \in I$ for a system satisfies

1. $S(\{p_i\})$ is a continuous function of $p_i,\ i \in I$
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3. $S(p_1, p_2, \ldots, p_n, 0) = S(p_1, p_2, \ldots, p_n)$
4. If A, B are subsystems, their combination has joint probability distribution $\{p_{ij},\ i \in I_A,\ j \in I_B\}$, marginals $p_i(A) = \sum_{j \in I_B} p_{ij},\ p_j(B) = \sum_{i \in I_A} p_{ij}$, then

$$\frac{1}{k_B} S(\{p_{ij}\}) \ = \ \frac{1}{k_B} S(\{p_i\}) + \frac{1}{k_B} S(\{p_j|p_i\}) + \frac{1}{k_B^2}(1-q) S(\{p_i\}) S(\{p_j|p_i\})$$

with $S(\{p_j|p_i\}) = \frac{\sum_{i \in I_A} p_i^q S(\{p_{ij}/p_i\})}{\sum_{i \in I_A} p_I^q}$ conditional escort prob. of B given A

Then $$S \ = \ -k \, \frac{1}{q-1} \left\{ 1 - \sum_{i \in I} p_i^q \right\} \qquad \text{(with } \ k > 0)$$

10. Dynamical and Physical Foundations of S_q

- Not much is known.
- Analytical, model-independent arguments are very few, far apart and circumstantial, at best.
- Conjecturally S_q describes non-ergodic systems, systems with strong spatial and temporal correlations, long-range interactions etc.
- Systems whose phase space volume increases polynomially (Hanel-Thurner 2011).
- Systems with largest Lyapunov exponent zero: weak chaos. Re-define meaningful "Lyapunov-like" exponents for such cases.
- Systems with long-range interactions exhibit "anomalous" behaviour such as quasi-stationary states. How can we describe such systems?

11. Some Weaknesses (limitations?) of S_q

- A thermodynamic formalism mirroring that based on S_{BGS} has been, largely, developed for S_q. **However ...**
- Several points need to still be elucidated:
  - What is the meaning and physical interpretation of q ?
  - If one combines two systems with different q's, what will be the result?
  - How does one define temperature, heat exchange etc for different q's?
  - Does the first law of thermodynamics even apply to interacting systems with different q's?
  - Why use the "maximum entropy" principle for deriving the thermodynamic behaviour of such systems?
  - What is the origin and role of escort distributions in this framework?
  - Be **very careful** with data fittings and the emergence of q-exponentials.
  - Does S_q have any physical significance ?
    Calculate S_q for particular models and compare to experiments.

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THANK YOU