Dark Solitons: From 1D to 2D and 3D with Some Quantum Touches

Dynamical Systems/Complexity Summer School, Athens, July 17, 2018

P.G. Kevrekidis

University of Massachusetts

In Collaboration With:

- D.J. Frantzeskakis (Athens), R. Carretero (SDSU)
- W. Wang, E. Charalampidis (Amherst), J. D'Ambroise (SUNY OW)
- P. Schmelcher, L. Katsimiga, S. Mistakidis, G. Koutentakis (Hamburg)
- P. Engels, A. Khamechi, T. Bersano, V. Gokhroo (Washington State)
- R. Bisset, C. Ticknor, L. Collins (LANL)

with the gratefully acknowledged partial support of:

- US-NSF (DMS and PHY); Stavros Niarchos Foundation.
- Alexander von Humboldt Foundation; QNRF (Qatar).

References

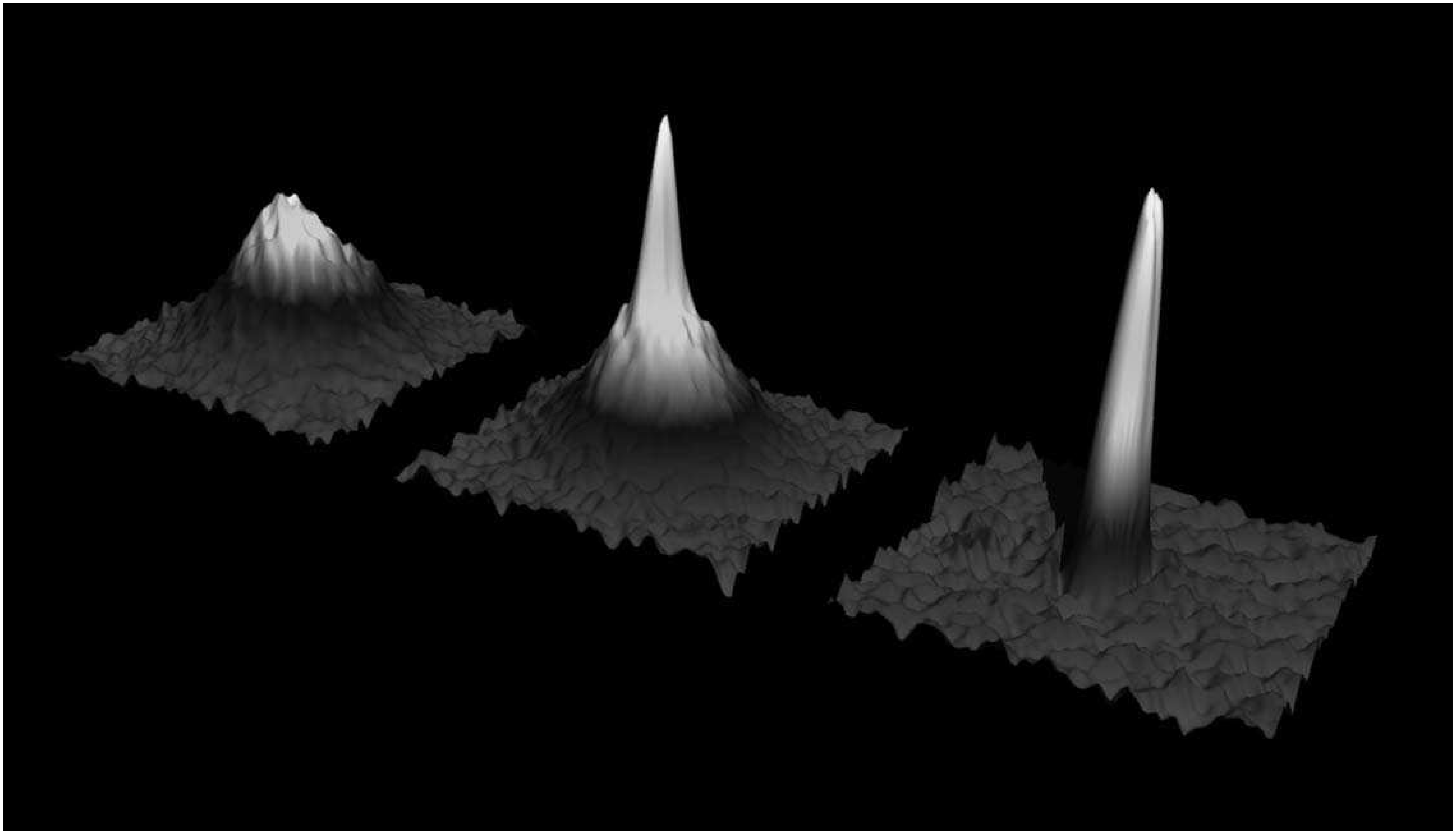
- PRL **118**, 244101 (2017);
- PRA **97**, 063604 (2018);
- PRL **120**, 063202 (2018);
- NJP **19**, 073004 (2017);
- NJP **19**, 123012 (2017).
- Recent Overviews:
  - Reviews in Physics **1**, 140 (2016)
  - Defocusing NLS Book, SIAM (OT 143).

Brief Introduction to BECs

- 1924: S. Bose and A. Einstein realize that Bose statistics predicts a Maximum Atom Number in the Excited States: a Quantum Phase Transition.

- 1995: E. Cornell, C. Wieman and W. Ketterle realize BEC in a dilute gas of ^{87}Rb and ^{23}Na: 2001 Nobel Prize.

- Today:
  - ~ 50 Experimental Groups have achieved BEC (in 100-10^8 atoms of Rb, Li, Na, H).
  - O(10^4) Theoretical and O(10^3) Experimental papers !



Mean-Field Models of BEC: why do we care ?

BEC

- Many Body Hamiltonian

$$\hat{H} = \int d\mathbf{r}\hat{\Psi}^{\dagger}\left[-\frac{\hbar^2}{2m}\Delta + V_{\text{ext}}(\mathbf{r})\right]\hat{\Psi} + \frac{1}{2}\int d\mathbf{r}d\mathbf{r}'\hat{\Psi}^{\dagger}(\mathbf{r})\hat{\Psi}^{\dagger}(\mathbf{r}')V(\mathbf{r}-\mathbf{r}')\hat{\Psi}(\mathbf{r}')\hat{\Psi}(\mathbf{r}) \quad (1)$$

- Bogoliubov Decomposition:

$$\hat{\Psi} = \Phi(\mathbf{r},t) + \hat{\Psi}'(\mathbf{r},t) \quad (2)$$

- Φ is now a regular wavefunction (the expectation value of the field operator). Its equation:

$$i\hbar\frac{\partial\Phi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Phi + V_{\text{ext}}(\mathbf{r})\Phi + g|\Phi|^2\Phi \quad (3)$$

- for dilute, cold, binary collision gas.
- But: This is 3D NLS with a Potential: GP !

Low Dimensional Reductions

- 1d Magnetic Trap and/or Optical Lattice

$$V(x) = \frac{1}{2}\Omega^2 x^2 + V_0 \sin^2(kx + \theta) \tag{4}$$

- 2d Magnetic Trap and/or Optical Lattice

$$V(x,y) = \frac{1}{2}\left(\Omega_x^2 x^2 + \Omega_y^2 y^2\right) + V_0\left(\sin^2(kx+\theta) + \sin^2(ky+\theta)\right) \tag{5}$$

- Typical 1d Scenario: $g > 0 \Rightarrow$ Exact Prototypical Solutions: Dark Solitons

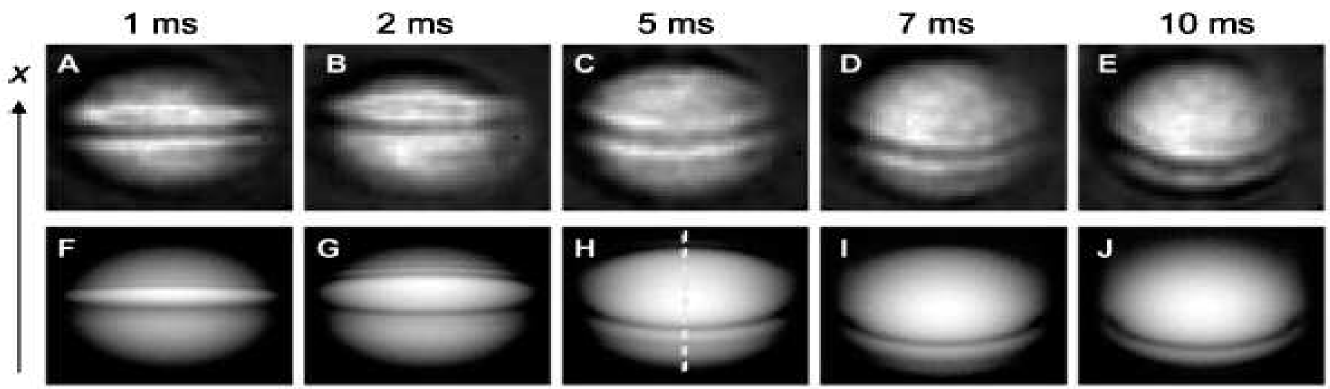
$$\Phi(x,t) = e^{-it}\tanh(x - x_0) \Rightarrow n = |\Phi|^2 = \tanh^2(x - x_0) \tag{6}$$

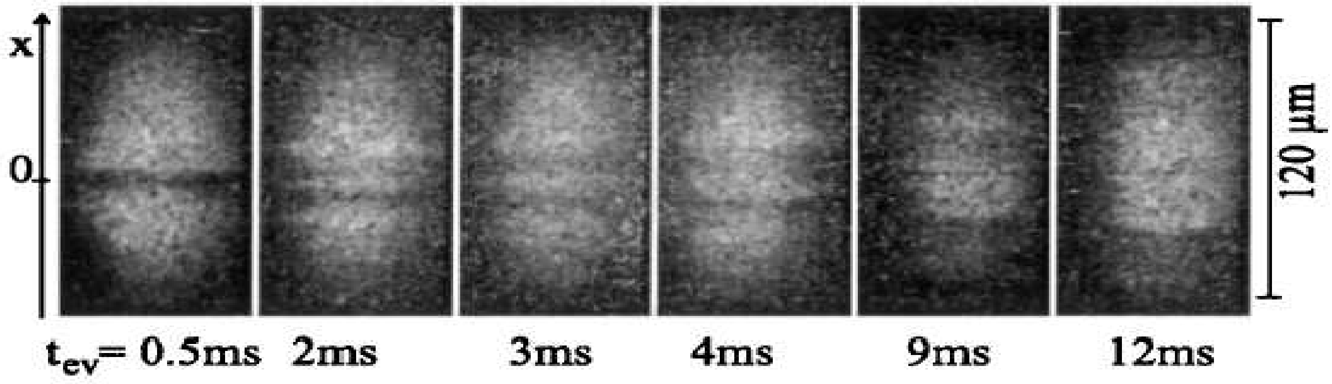
- It is also possible to have Multiple Spin States of a Bose Gas (such as ^{87}Rb or ^{23}Na or mixtures thereof) $\Rightarrow$ In this setting, the Vector NLS Model reads:

$$i\frac{\partial \psi_n}{\partial t} = -\frac{1}{2}\nabla^2 \psi_n + V_n(\mathbf{r})\psi_n + \sum_{k=1}^{\mathcal{N}} \left[g_{nk}|\psi_k|^2\psi_n - \kappa_{nk}\psi_k + \Delta_{nk}\psi_n\right]. \tag{7}$$

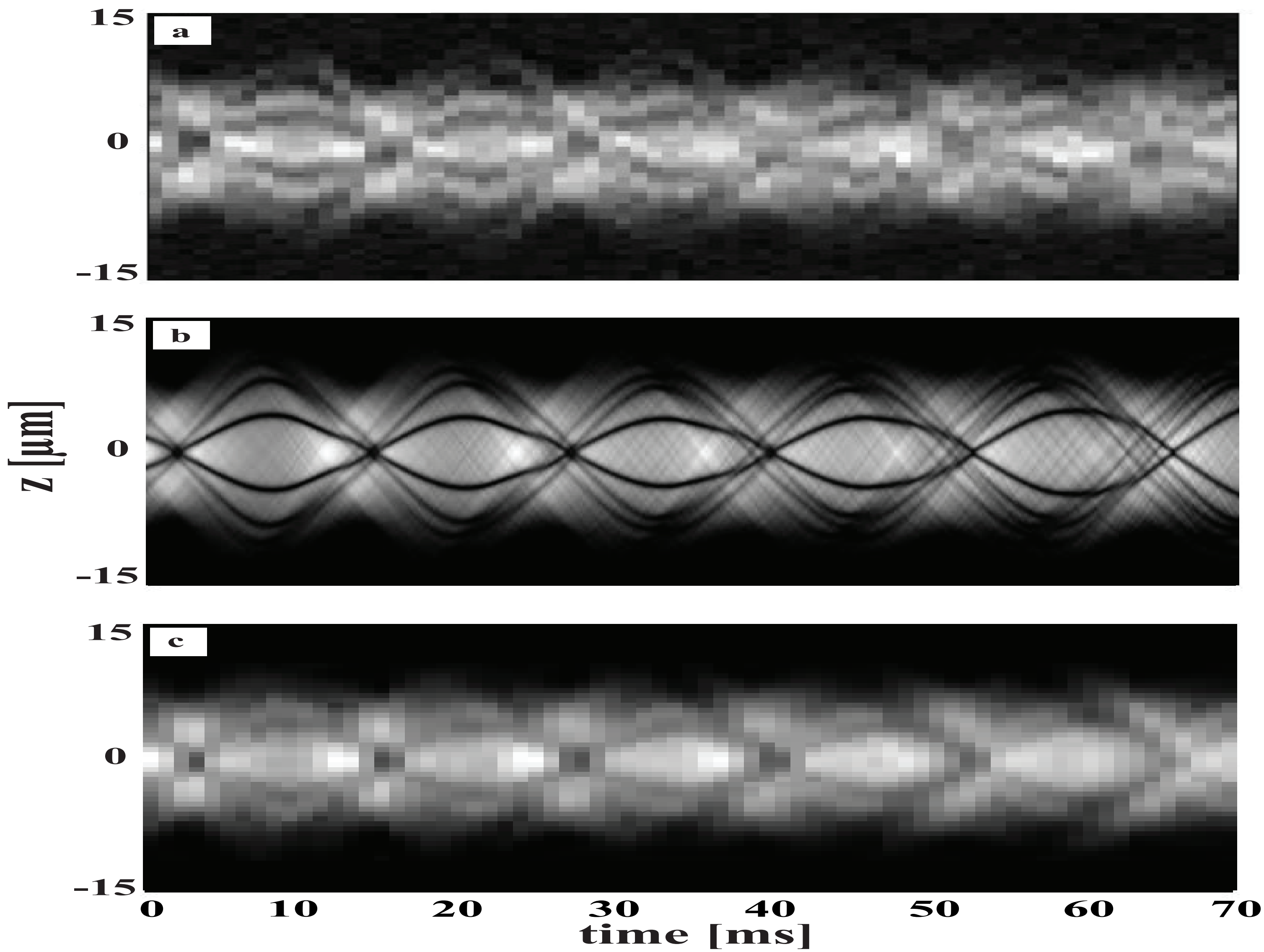
One Component Motivation: Dark Soliton Dynamics

Early Experiments in JILA, NIST, Hanover

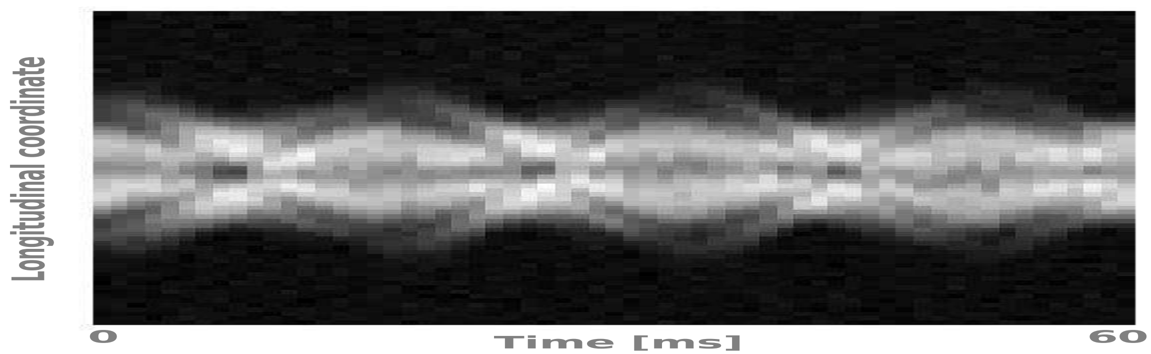


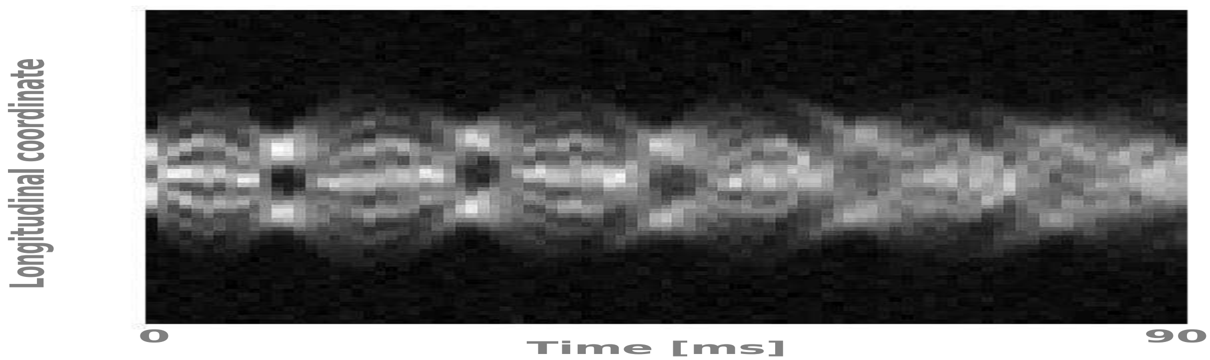


Improved Experiments in Heidelberg (M. Oberthaler group)

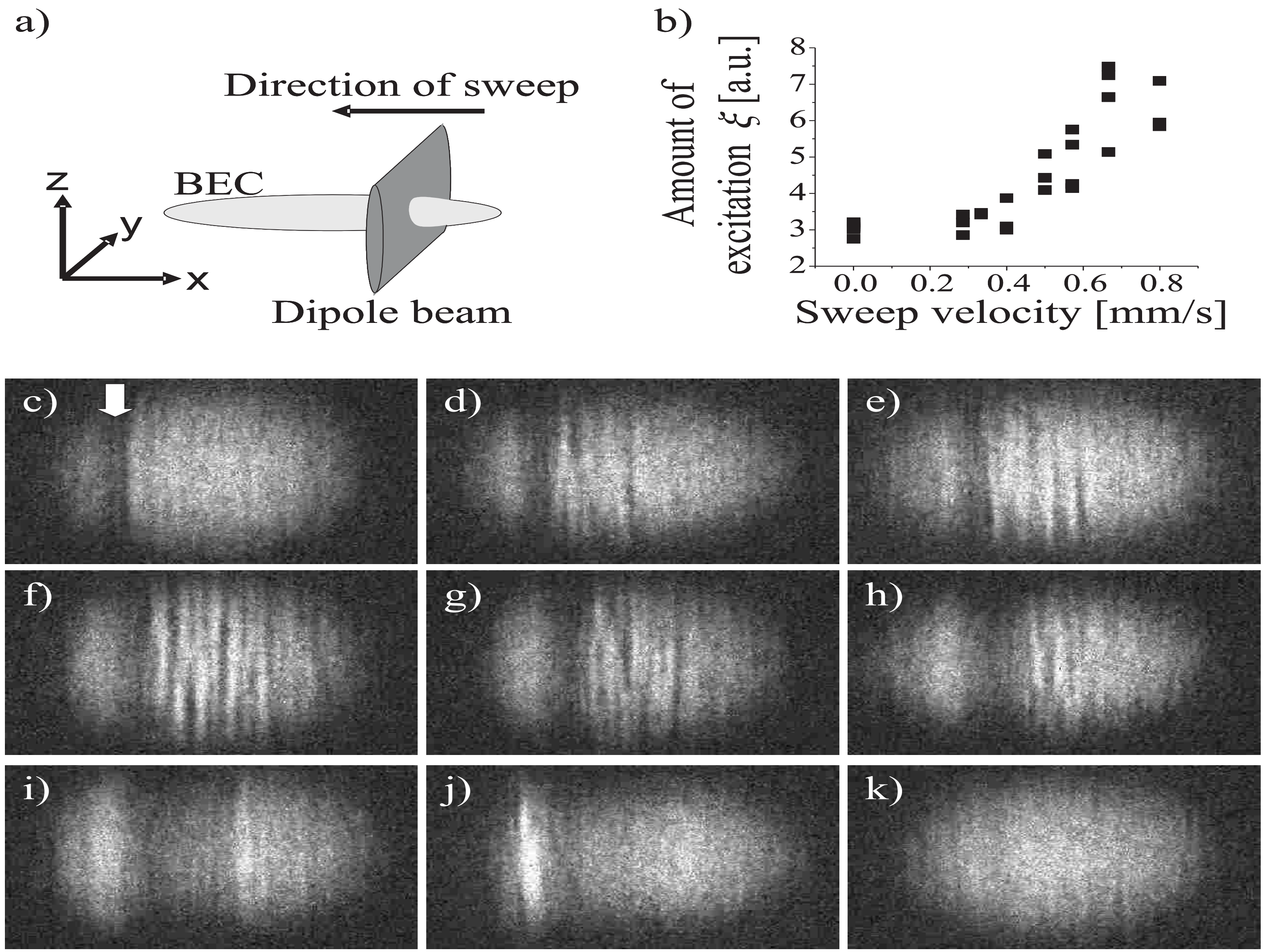


3-, 4-, N-soliton States

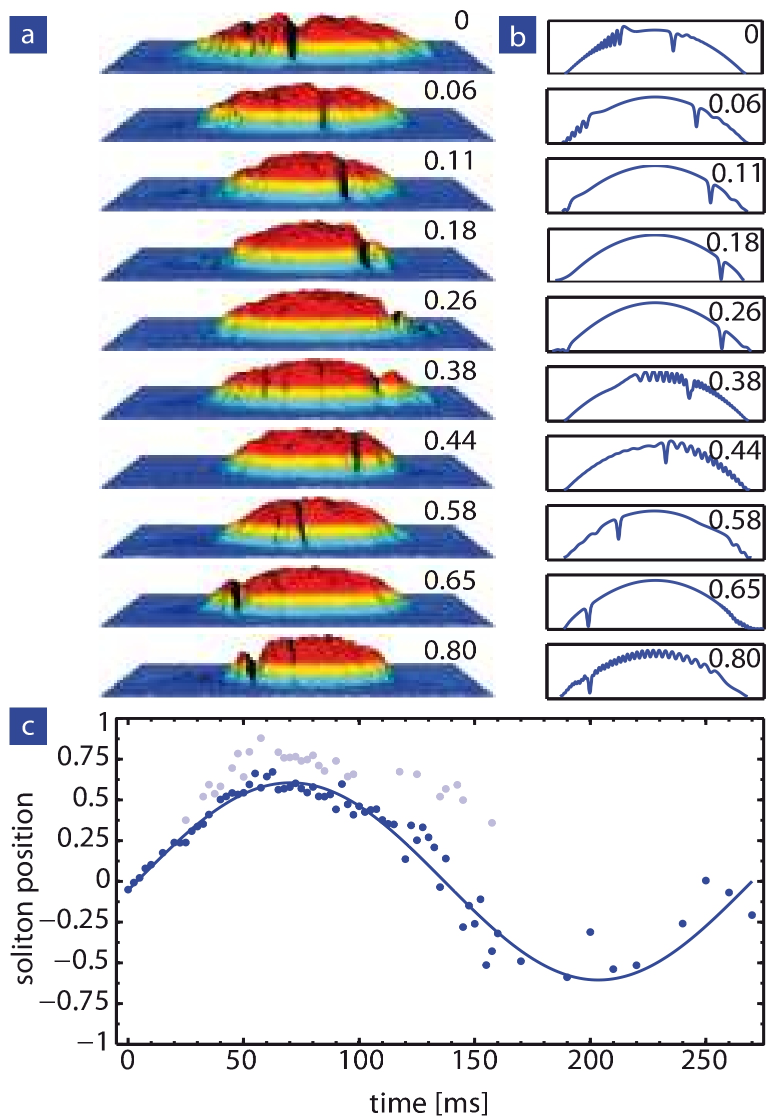


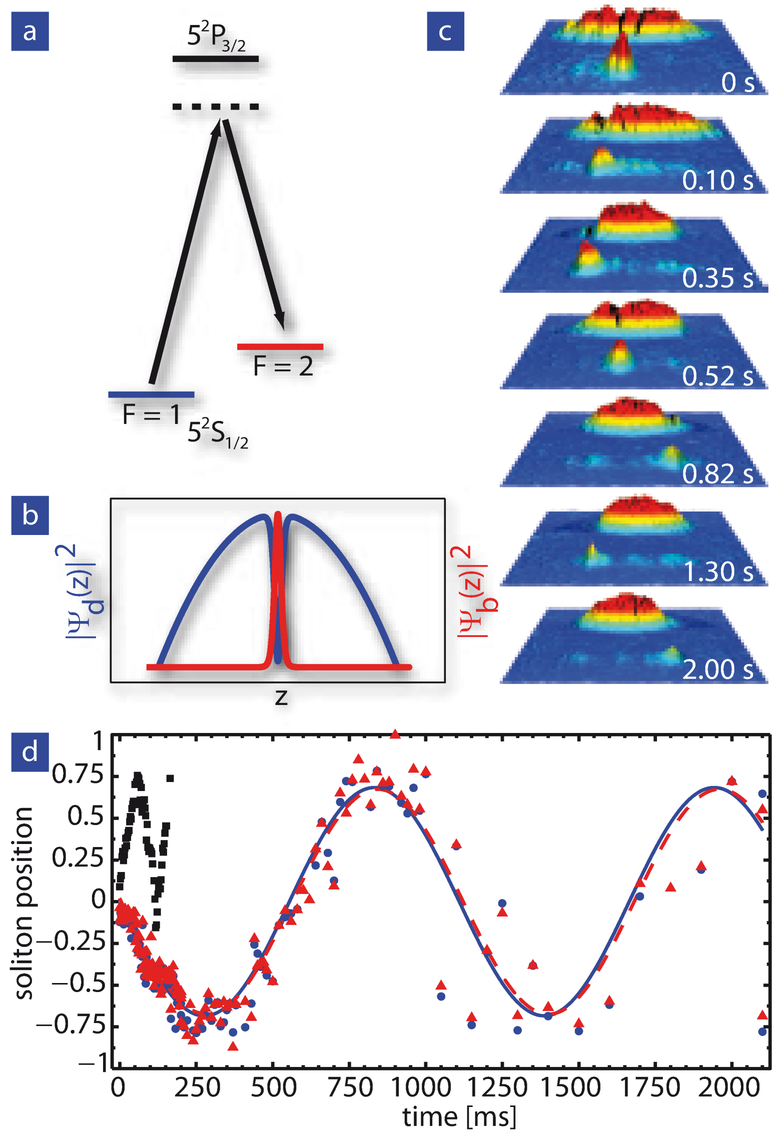


Improved Experiments in Pullman (P. Engels group)



Improved Experiments in Hamburg (K. Sengstock group)




a
$5^2P_{3/2}$
F = 2
F = 1
$5^2S_{1/2}$
b
$|\Psi_d(z)|^2$
$|\Psi_b(z)|^2$
z
c
0 s
0.10 s
0.35 s
0.52 s
0.82 s
1.30 s
2.00 s
d
soliton position
time [ms]
1
0.75
0.5
0.25
0
−0.25
−0.5
−0.75
−1
0
250
500
750
1000
1250
1500
1750
2000

Two-Component Motivation: Dark-Bright Solitons in Nonlinear Optics

- Dark-Bright Solitons were shown to Robustly Persist in Photorefractive Crystals

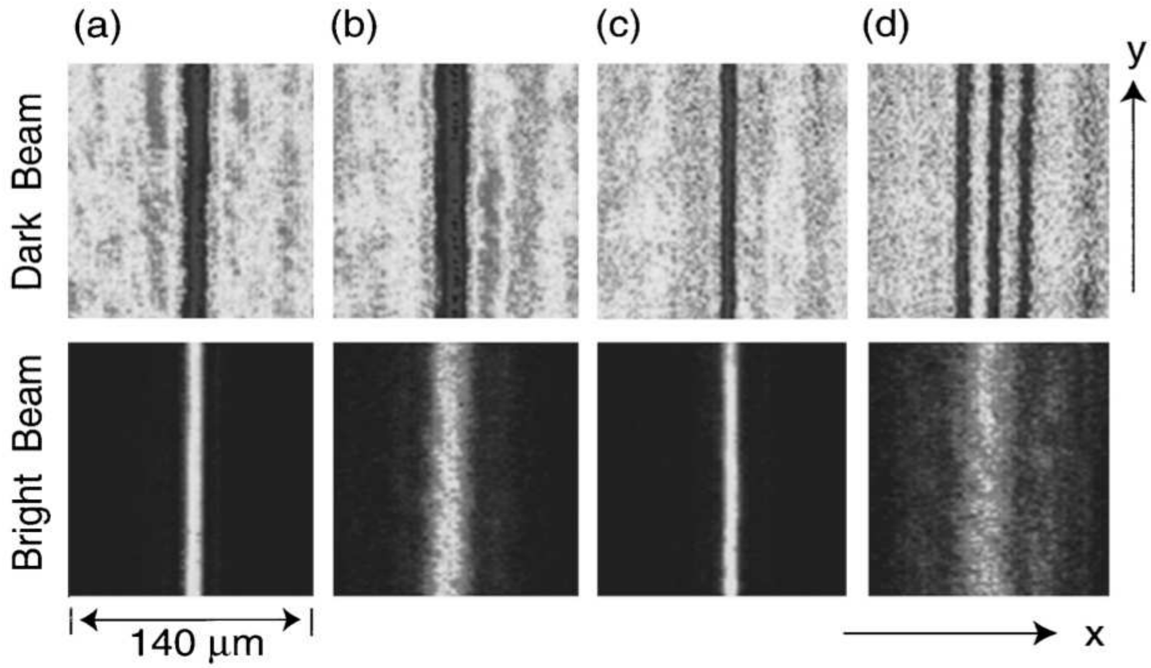


Fig. 7

Citation
Zhigang Chen, Mordechai Segev, Tamer H. Coskun, Demetrios N. Christodoulides, Yuri S. Kivshar, "Coupled photorefractive spatial-soliton pairs," J. Opt. Soc. Am. B **14,** 3066-3077 (1997);
http://www.opticsinfobase.org/josab/abstract.cfm?URI=josab-14-11-3066

Further Early Motivation: Dark-Bright Soliton Pairs in Photorefractives

- Optical (Dark) Solitons were found to be Glued Together by Attraction between the Non-Soliton Beams they Guide
- This gave rise to the notion of Solitonic Gluons

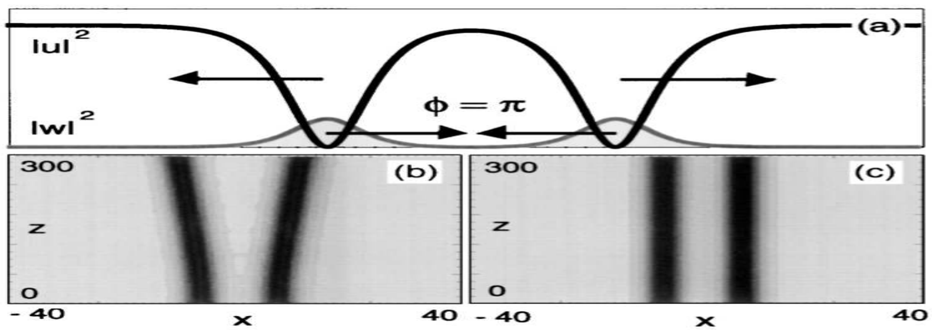


Fig. 1

Citation
Elena A. Ostrovskaya, YuriS. Kivshar, Zhigang Chen, Mordechai Segev, "Interaction between vector solitons and solitonic gluons," Opt. Lett. **24**, 327-329 (1999);
http://www.opticsinfobase.org/ol/abstract.cfm?URI=ol-24-5-327

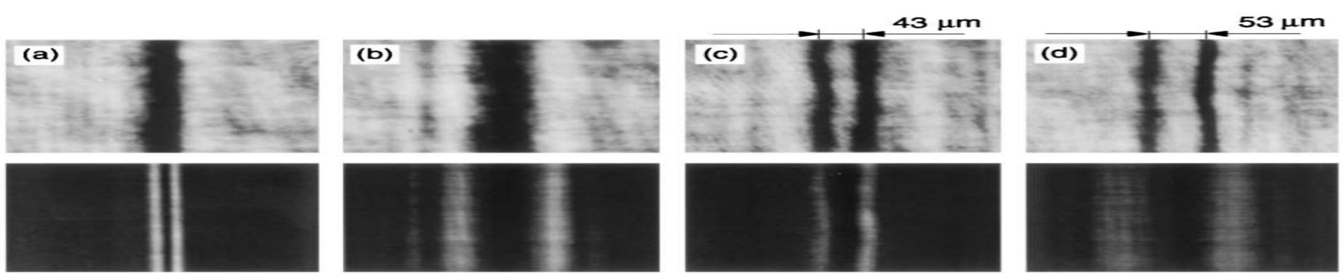
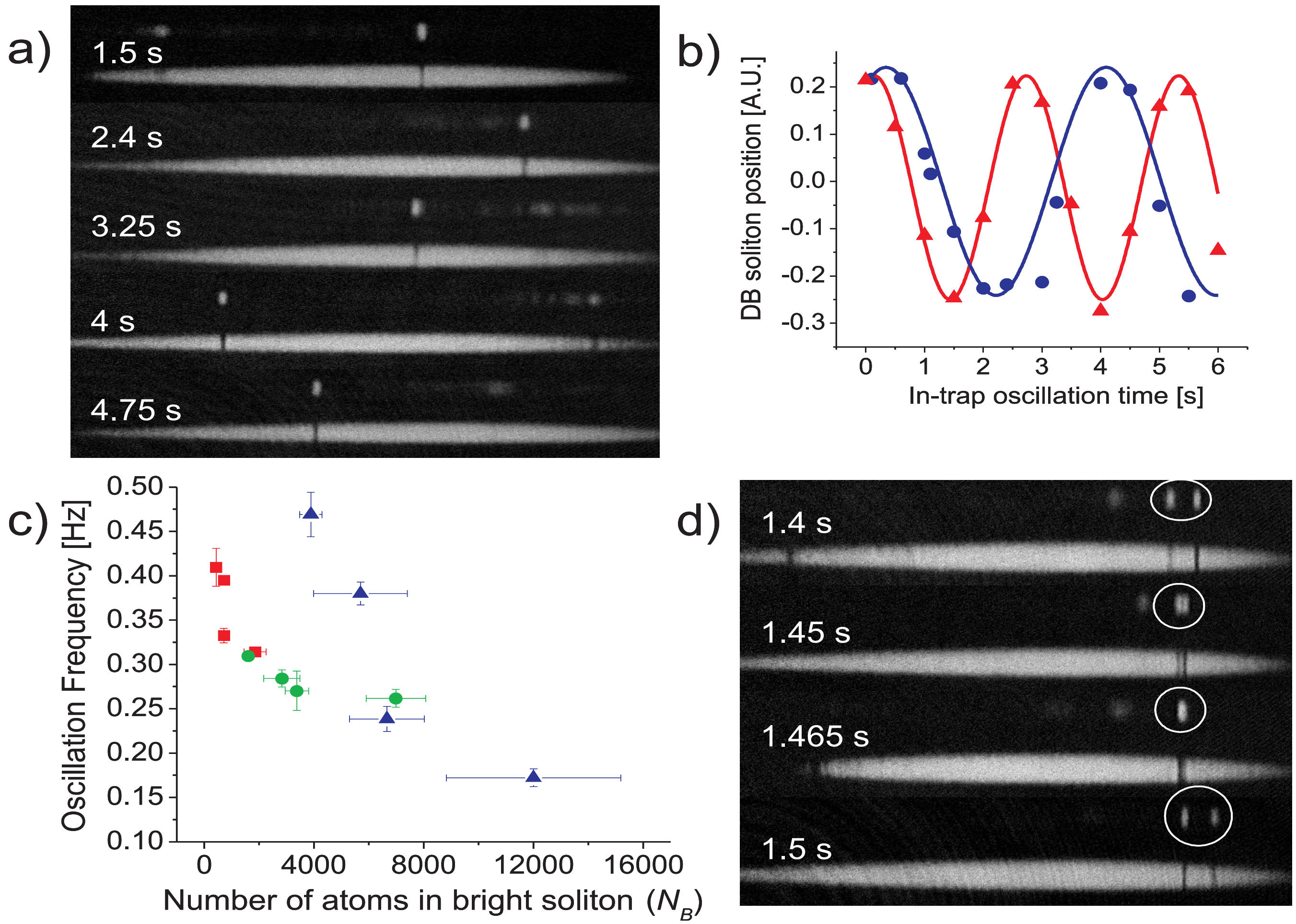


Fig. 3

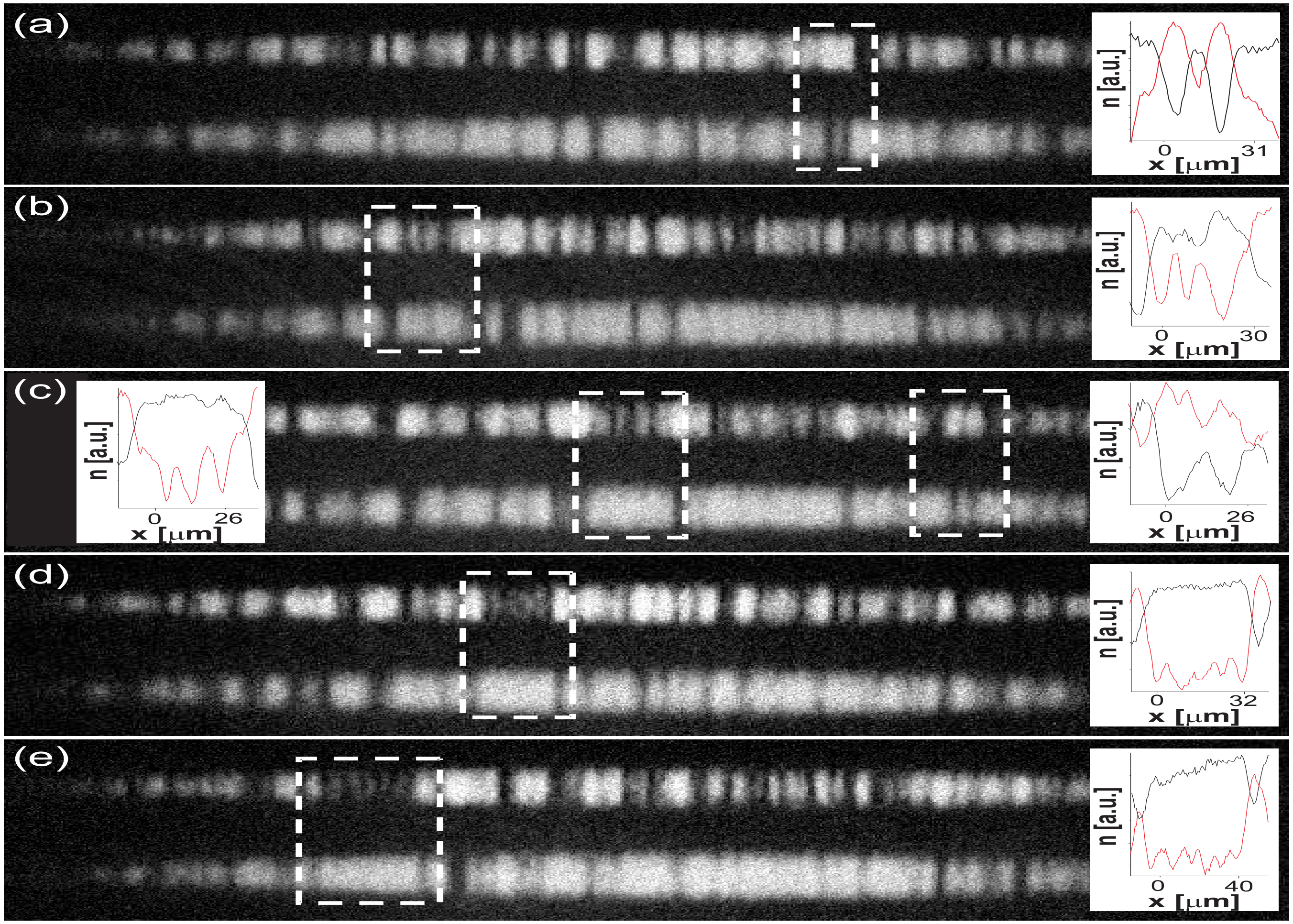
Citation
Elena A. Ostrovskaya, YuriS. Kivshar, Zhigang Chen, Mordechai Segev, "Interaction between vector solitons and solitonic gluons," Opt. Lett. **24**, 327-329 (1999);
http://www.opticsinfobase.org/ol/abstract.cfm?URI=ol-24-5-327

More Recent Motivation: (Pseudo)-Spinor Experiments in BECs

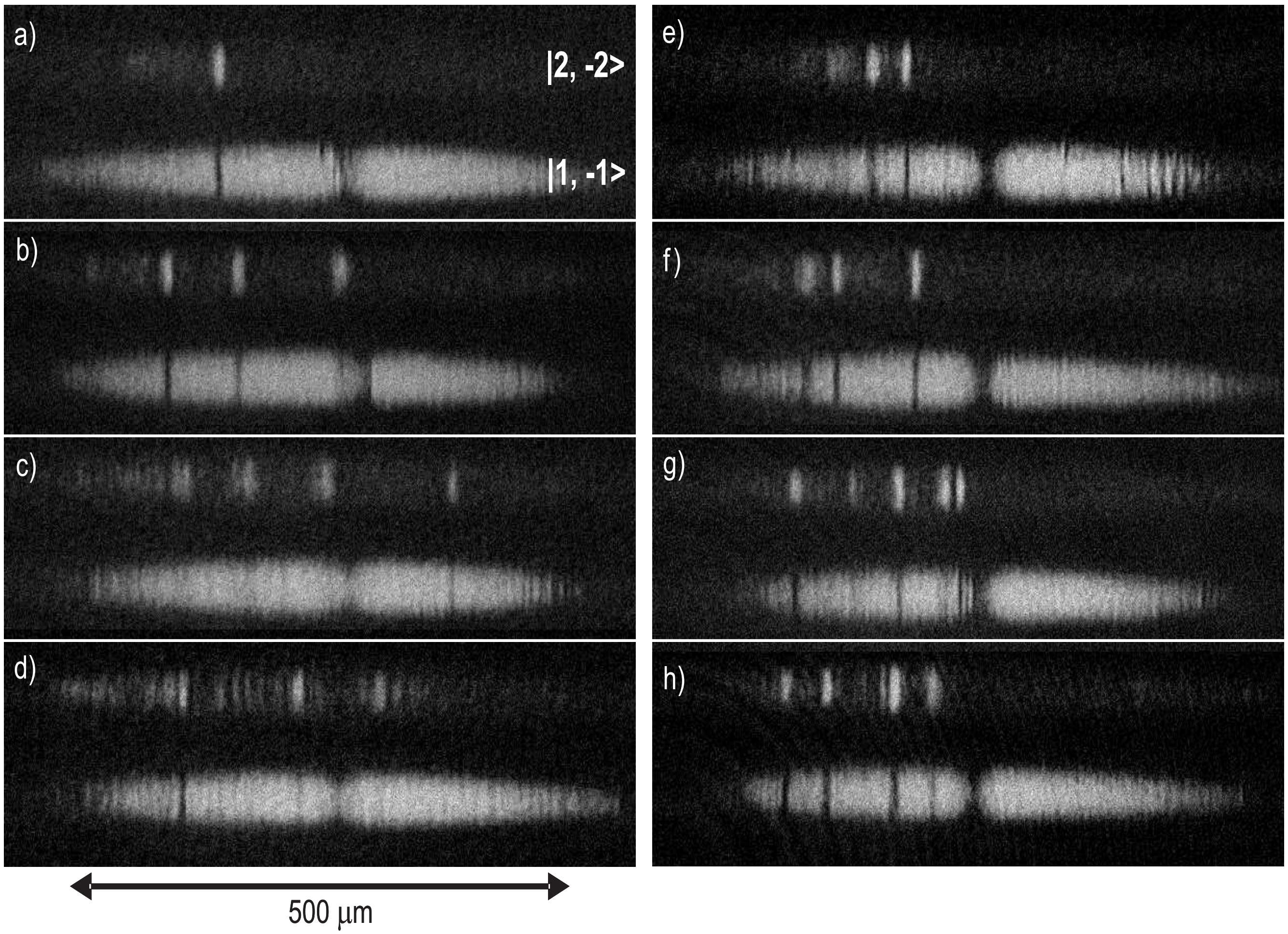
2-Components, 1-dimension: Dark-Bright Solitons in Pullman



More Complex Configurations: Multi-Dark-Bright Solitons in BECs (2, 3, 4, 5,...)

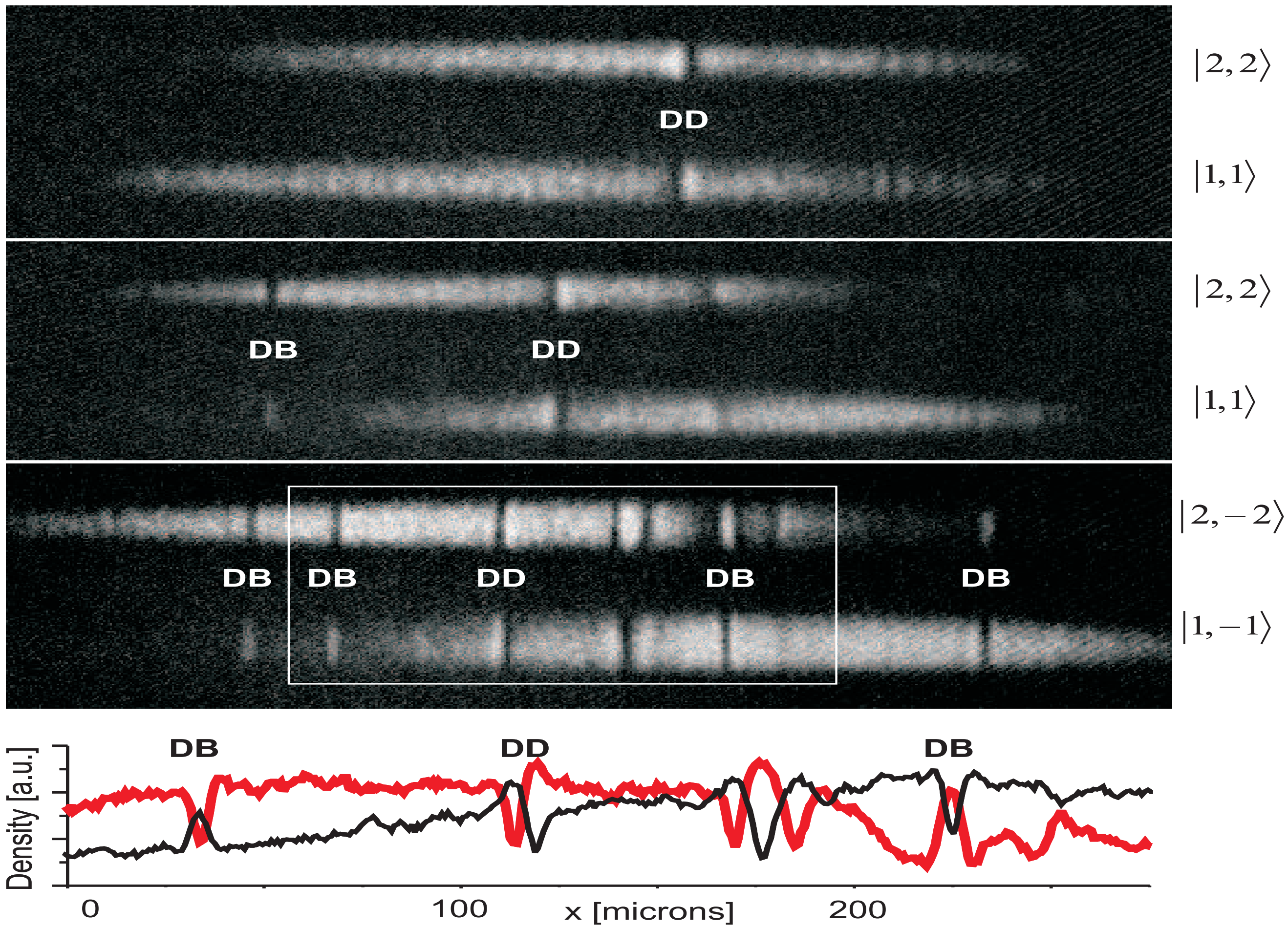


More Complex Dynamics: Interaction of Dark-Bright Solitons with Barriers

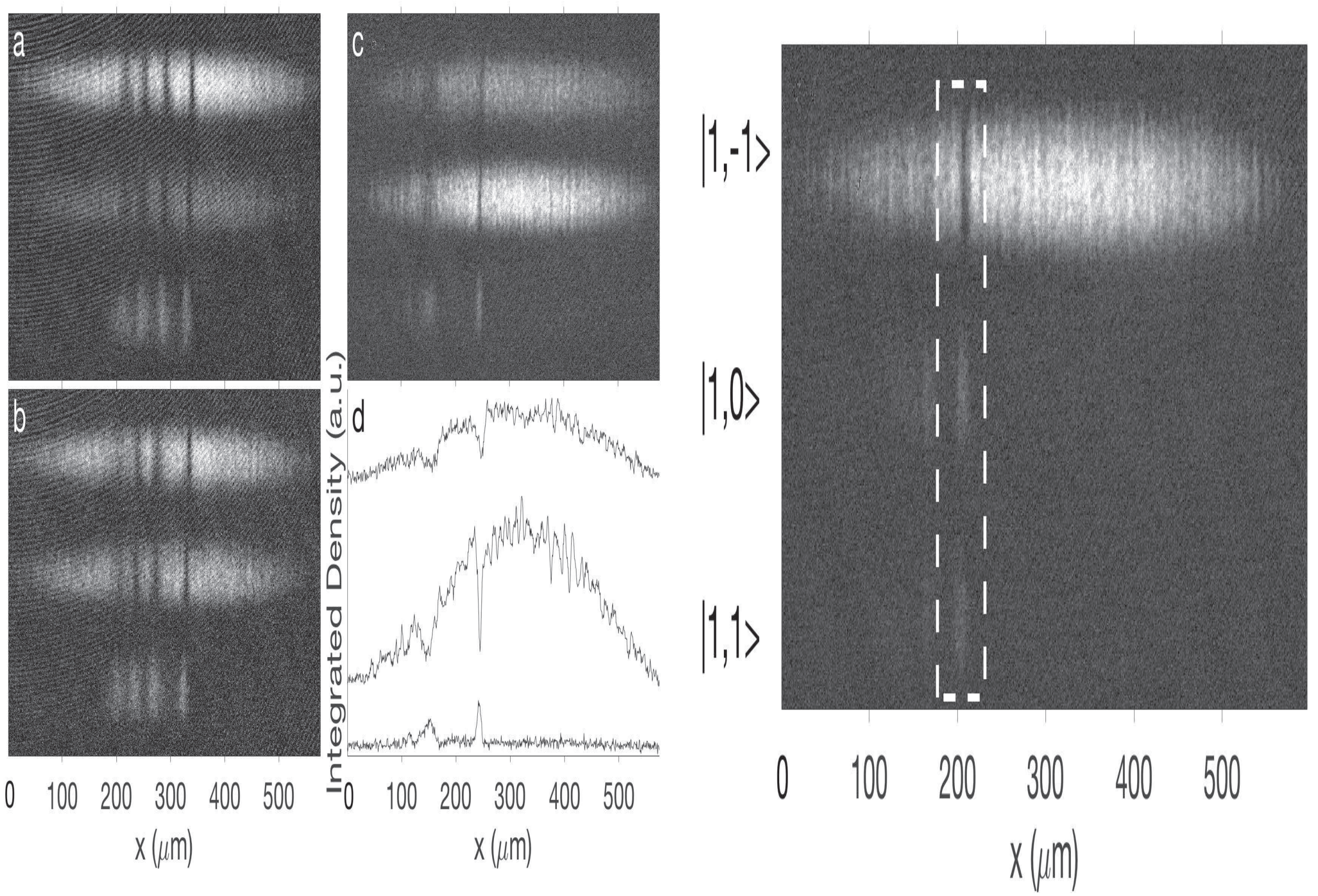


(Even) More Complex Dynamics: Counter-Flow Experiments

Spontaneous Production of Dark-Bright and Dark-Dark Solitons



Another Recent Addition: Creation of DBB and DDB in Experiments



One Component Analysis: the Near-Linear Limit

- The Simplest Model reads

$$iU_t + \frac{1}{2}U_{xx} - |U|^2 U = \frac{1}{2}\omega^2 x^2 U \qquad (8)$$

- Model assumes strong anisotropy ($\omega \ll 1$), and $\mu \ll \hbar\omega_\perp$, so that $\phi_0(r) \propto \exp(-r^2/2a_r^2)$.

- The Steady State Problem, for $U(x,t) = e^{-i\mu t}u(x)$ (μ is the Chemical Potential) with $\mathcal{L} = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}\omega^2 x^2$ reads:

$$\mu u = \mathcal{L}u + |u|^2 u \qquad (9)$$

- Consider Expansion near the Linear Limit $u = \sqrt{\epsilon}u_0 + \epsilon^{3/2}u_1 + \ldots$ and $\mu = \mu_0 + \epsilon\mu_1 + \ldots$ This leads to the solvability condition $\mu_1 = \int |u_0|^4 dxdydz$.

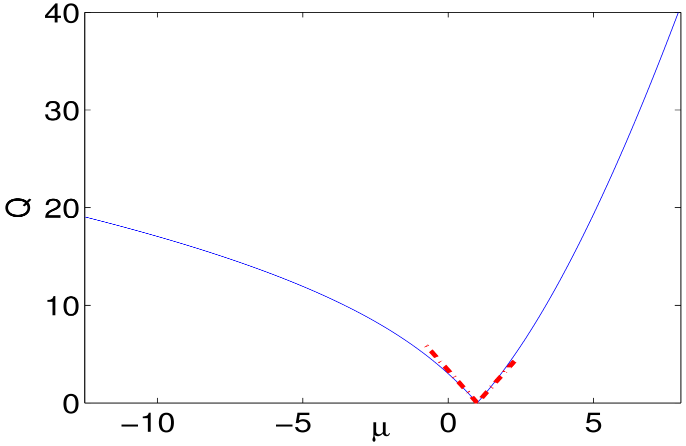
- The Linearization Bogoliubov-de Gennes problems then reads:

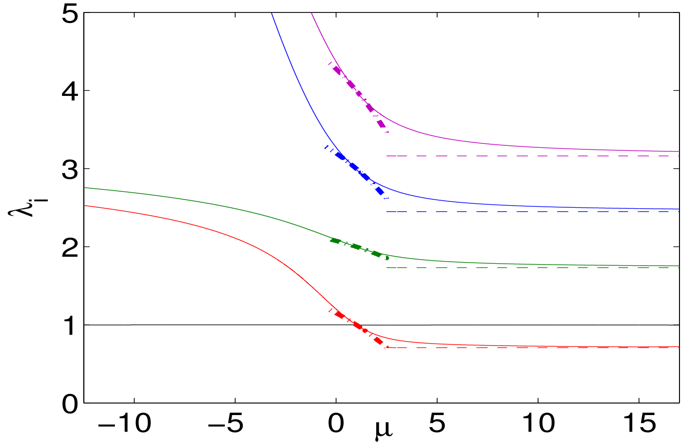
$$\mathcal{H}_0 = \begin{pmatrix} \mathcal{L}_1 & 0 \\ 0 & -\mathcal{L}_1 \end{pmatrix}, \qquad (10)$$

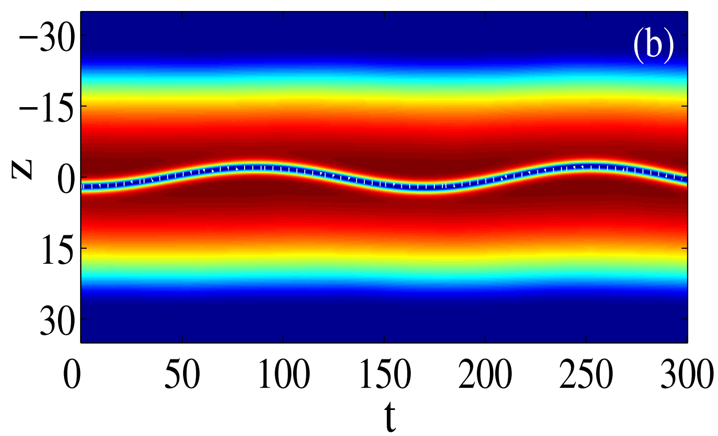
where $\mathcal{L}_1 = \mathcal{L} - \mu_0$ while

$$\mathcal{H}_1 = \begin{pmatrix} 2|u_0|^2 - \mu_1 & u_0^2 \\ -(u_0^2)^\star & \mu_1 - 2|u_0|^2 \end{pmatrix}, \qquad (11)$$

Numerical Findings in 1d Case







- In the Linear Limit, the spectrum for 1, 2, 3, ... Dark Solitons is, respectively:

$$\Omega = \pm(n-1), \quad n-2, \quad n-3\ldots, \quad n = 0, 1, \ldots \tag{12}$$

- Using Perturbation Theory, for 1 Dark Soliton we obtain:

$$\left|\Omega_1 - 1 + \frac{\varepsilon^2}{8\sqrt{2\pi}}\right| \leq C_1 \varepsilon^4 \tag{13}$$

- Another Limit known is the so-called Thomas-Fermi Limit

$$\Omega_0 = 1, \;\; \lim_{\mu\to\infty} \Omega_1 = \frac{1}{\sqrt{2}}, \;\; \lim_{\mu\to\infty} \Omega_m = \frac{\sqrt{m(m+1)}}{\sqrt{2}}, \qquad m \geq 2 \tag{14}$$

- Dipolar Oscillation Frequency $\Omega_0 = 1$ is fixed due to Transformation

$$u(x,t) = e^{ip(t)x - \frac{i}{2}p(t)s(t) - \frac{i}{2}t - i\mu t - i\theta_0}\phi(x - s(t)), \tag{15}$$

where $\dot{s} = p$, $\dot{p} = -s$

Main Focus: Highly Nonlinear Limit

- Consider the GPE Energy

$$H_{1D} = \frac{1}{2}\int_{-\infty}^{\infty} |u_x|^2 + \left(|u|^2 - \mu\right)^2 dx.$$

- For a Dark Soliton Solution:

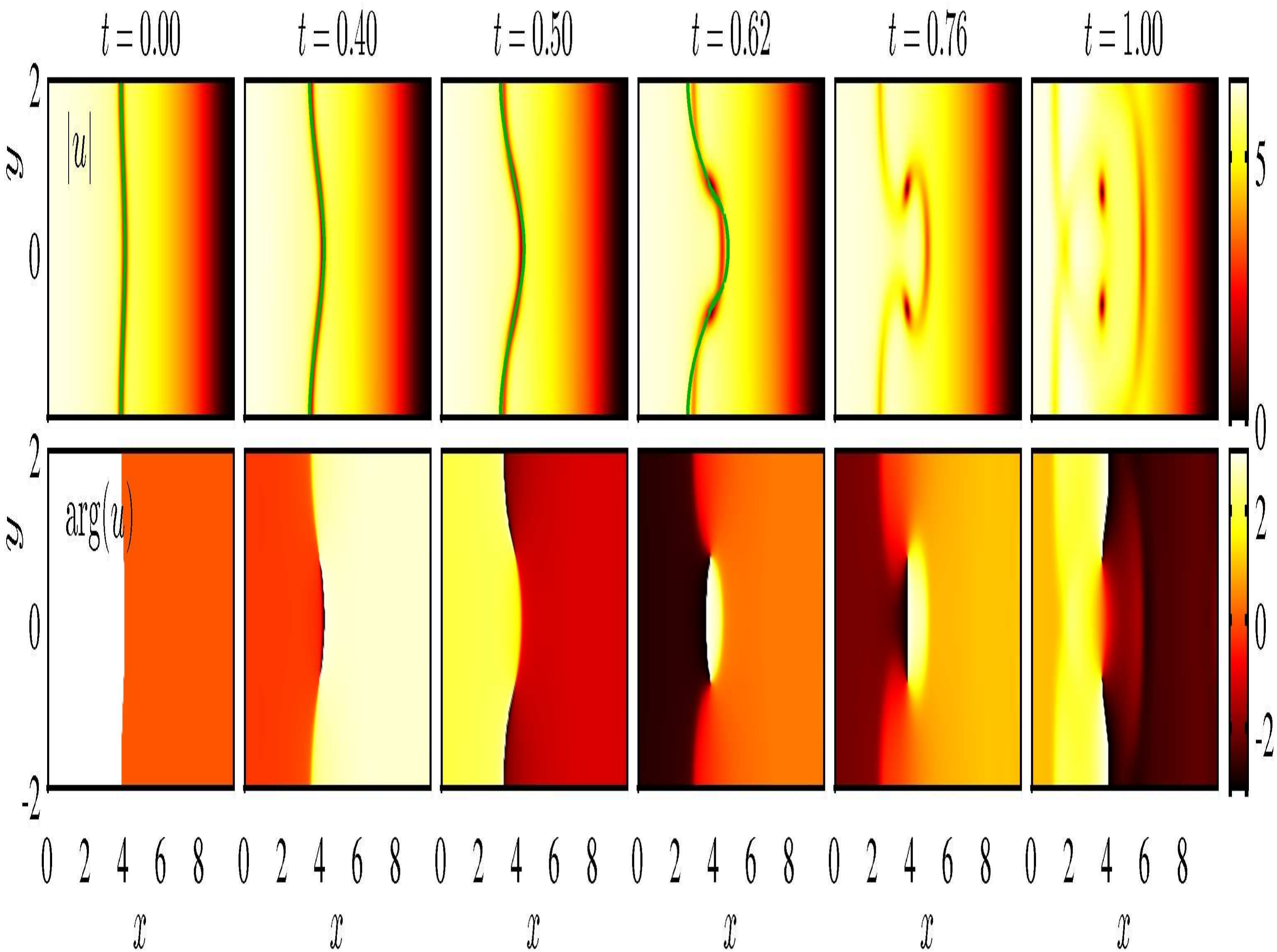
$$u(x,t) = e^{-i\mu t}\left[\sqrt{\mu - v^2}\,\tanh\left(\sqrt{\mu - v^2}(x - x_0)\right) + iv\right], \qquad (16)$$

- Obtain $H_{1D} = (4/3)(\mu - \dot{x}_0^2)^{3/2}$. Konotop-Pitaevskii (PRL, 2004) assuming the Adiabatic Invariance of $\mu \to \mu - V(x)$, obtained:

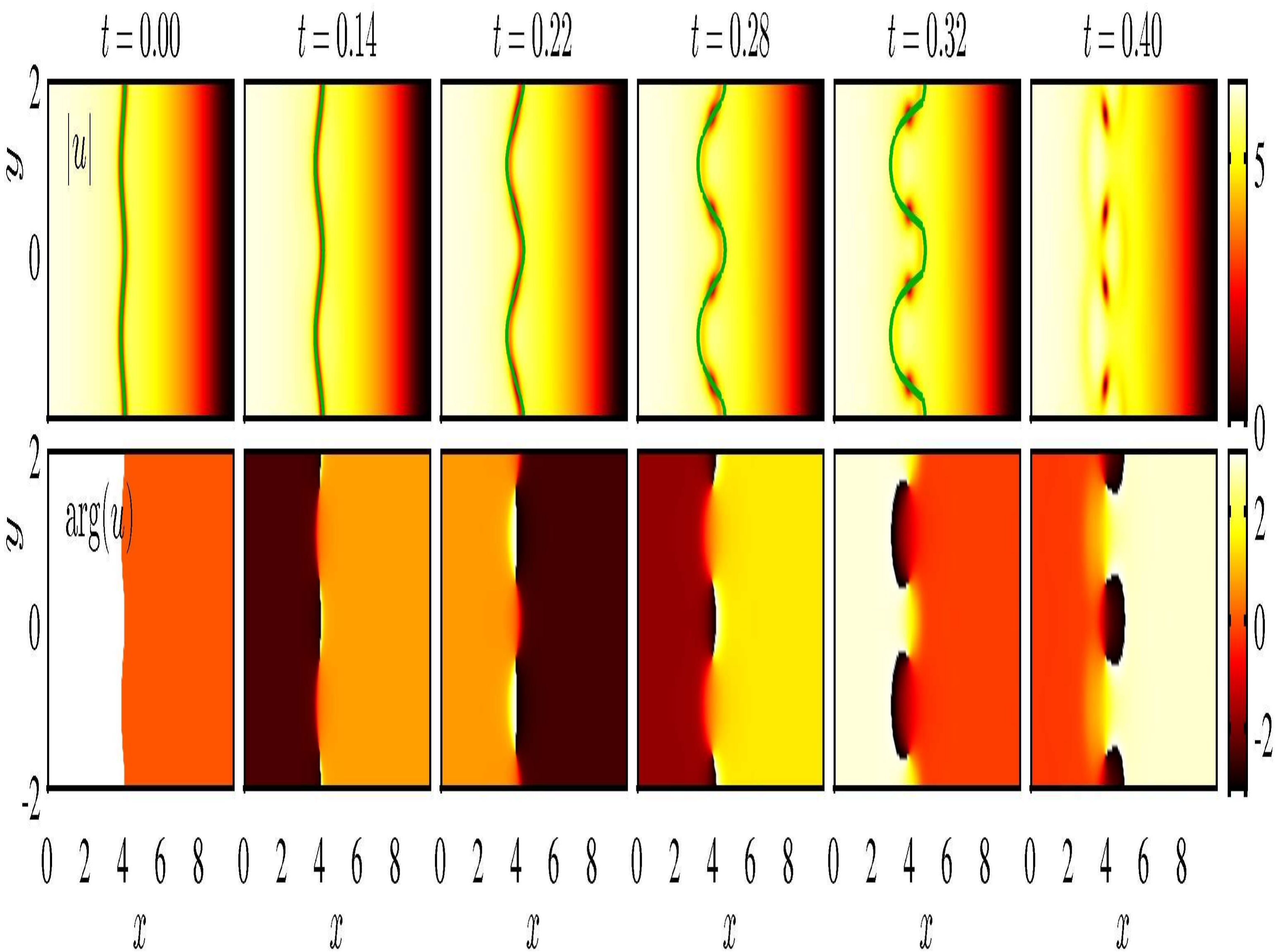
$$H_{1D} = \frac{4}{3}\left(\mu - V(x_0) - \dot{x}_0^2\right)^{3/2} \quad \Rightarrow \quad \ddot{x}_0 = -\frac{1}{2}V'(x_0), \qquad (17)$$

- Oscillatory Dynamics with $\omega = \frac{\Omega}{\sqrt{2}}$ for Parabolic Potential $V(x) = \frac{1}{2}\Omega^2 x^2$.

Higher Dimensional Case: Transverse Instability



Higher Dimensional Case: Transverse Instability (Contd.)



Theory: Adiabatic Invariants for Soliton Filaments

- Start with 2D GPE:

$$iu_t = -\frac{1}{2}(u_{xx} + u_{yy}) + |u|^2 u + V(x)u, \tag{18}$$

- Consider a 1D Potential and its center depending as $x_0 \to \xi = \xi(y,t)$, using the soliton in the 2D energy:

$$H_{2\text{D}} = \frac{1}{2}\iint_{-\infty}^{\infty} \left[|u_x|^2 + |u_y|^2 + \left(|u|^2 - \mu\right)^2\right] dx\, dy.$$

- Now, using the Solitonic Ansatz, we obtain the Filament Energy Functional:

$$E = \frac{4}{3}\int_{-\infty}^{\infty} \left(1 + \frac{1}{2}\xi_y^2\right) \left(\mu - V(\xi) - \xi_t^2\right)^{3/2} dy. \tag{19}$$

- From this, we can obtain the Filament Dynamical PDE with $A = \mu - V(\xi) - \xi_t^2$ and $B = 1 + \frac{1}{2}\xi_y^2$.

$$\xi_{tt}B + \frac{1}{3}\xi_{yy}A = \xi_y\, \xi_t\, \xi_{yt} - \frac{1}{2}V'(\xi)\left(B - \xi_y^2\right), \tag{20}$$

Theory: Adiabatic Invariants for Soliton Filaments (Contd.)

- Assuming that $\xi = \xi(t)$ is only a Function of Time yields

$$\xi_{tt} = -\frac{1}{2}V'(\xi),$$

- For Weak undulations, and for $V(x) = 0$, the dynamics is described by (cf. with Kuznetsov-Turitsyn (JETP, 1988))

$$\xi_{tt} + \frac{1}{3}\mu\,\xi_{yy} = 0,$$

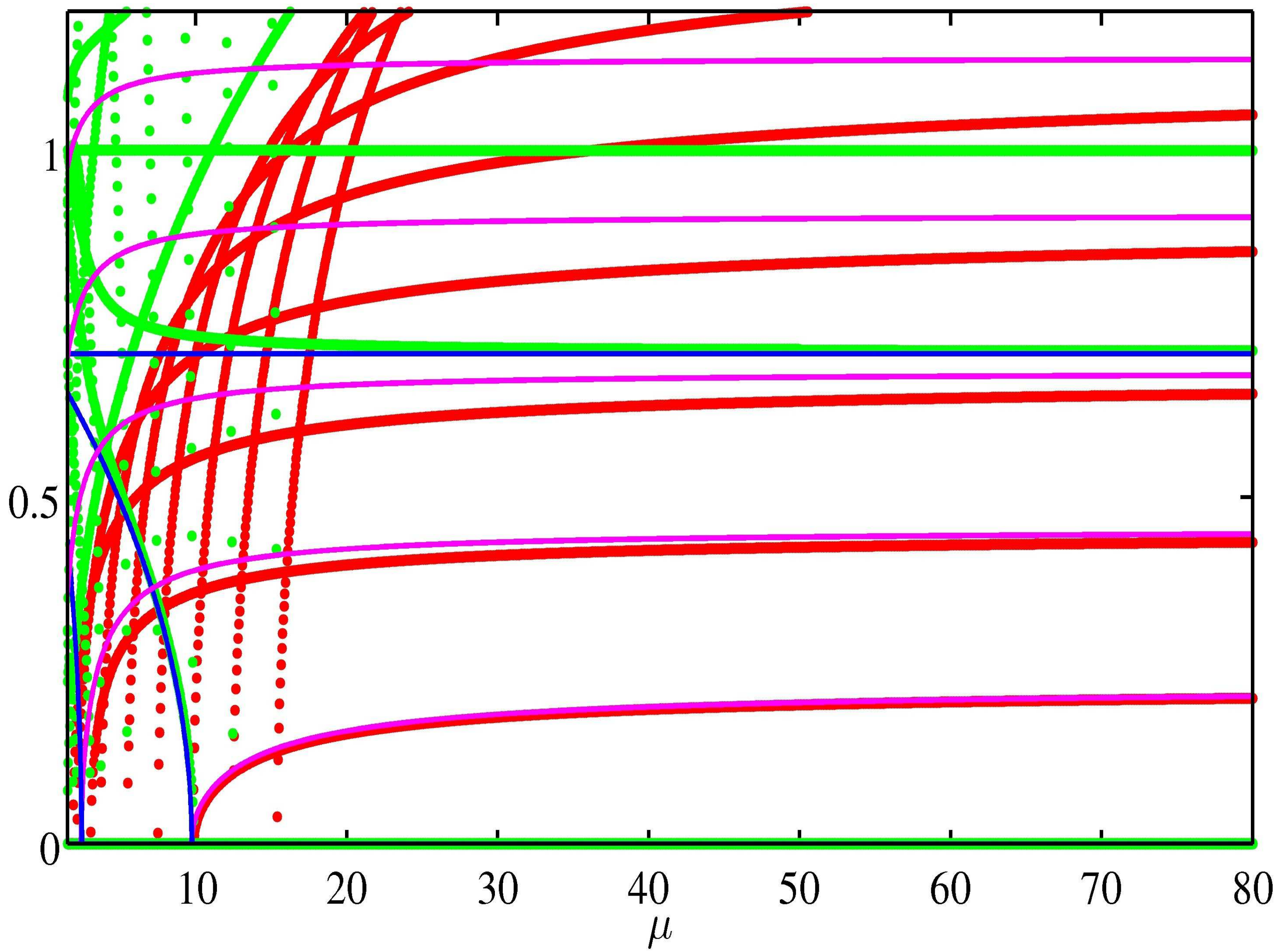
- For Weak undulations, and for $V \neq 0$, the Linearized PDE reads [this has implications for TI in Finite and Infinite Domains]:

$$\xi_{tt} + \frac{1}{3}(\mu - V(\xi_0))\,\xi_{yy} = -\frac{V''(\xi_0)}{2}\xi,$$

- For $V(x) = \frac{1}{2}\Omega^2 x^2$, one can Linearize Around a Uniform Equilibrium, using: $\xi(y,t) = X_0 + \epsilon \exp(\lambda t) \cos(k_n y)$ to obtain:

$$\lambda = i\omega = \sqrt{\frac{1}{3}\mu k_n^2 - \frac{1}{2}\Omega^2}, \tag{21}$$

Spectral Comparison



Adiabatic Invariants for Ring Dark Solitons

- For a Ring Dark Soliton in 2D, the Radial Energy is approximately: $E = 2\pi R \times (\mu - \dot{R}^2 - V(R))^{3/2}$.
- Including Azimuthal Undulations $R = R(\theta, t)$, we can obtain the Adiabatic Invariant

$$E = \frac{4}{3}\int_0^{2\pi} R\left(1 + \frac{R_\theta^2}{2R^2}\right)\left(\mu - R_t^2 - V(R)\right)^{3/2} d\theta. \qquad (22)$$

- From this, the Dynamically Relevant PDE Model with $C = \mu - V(R) - R_t^2$ and $D = 1 + R_\theta^2/(2R^2)$ reads:

$$CD - \frac{R_{\theta\theta}}{R}C = -\frac{R_\theta}{R}\left(\frac{3}{2}V'(R)R_\theta + 3R_tR_{t\theta}\right) + RD\left(\frac{3}{2}V'(R) + 3R_{tt}\right). \qquad (23)$$

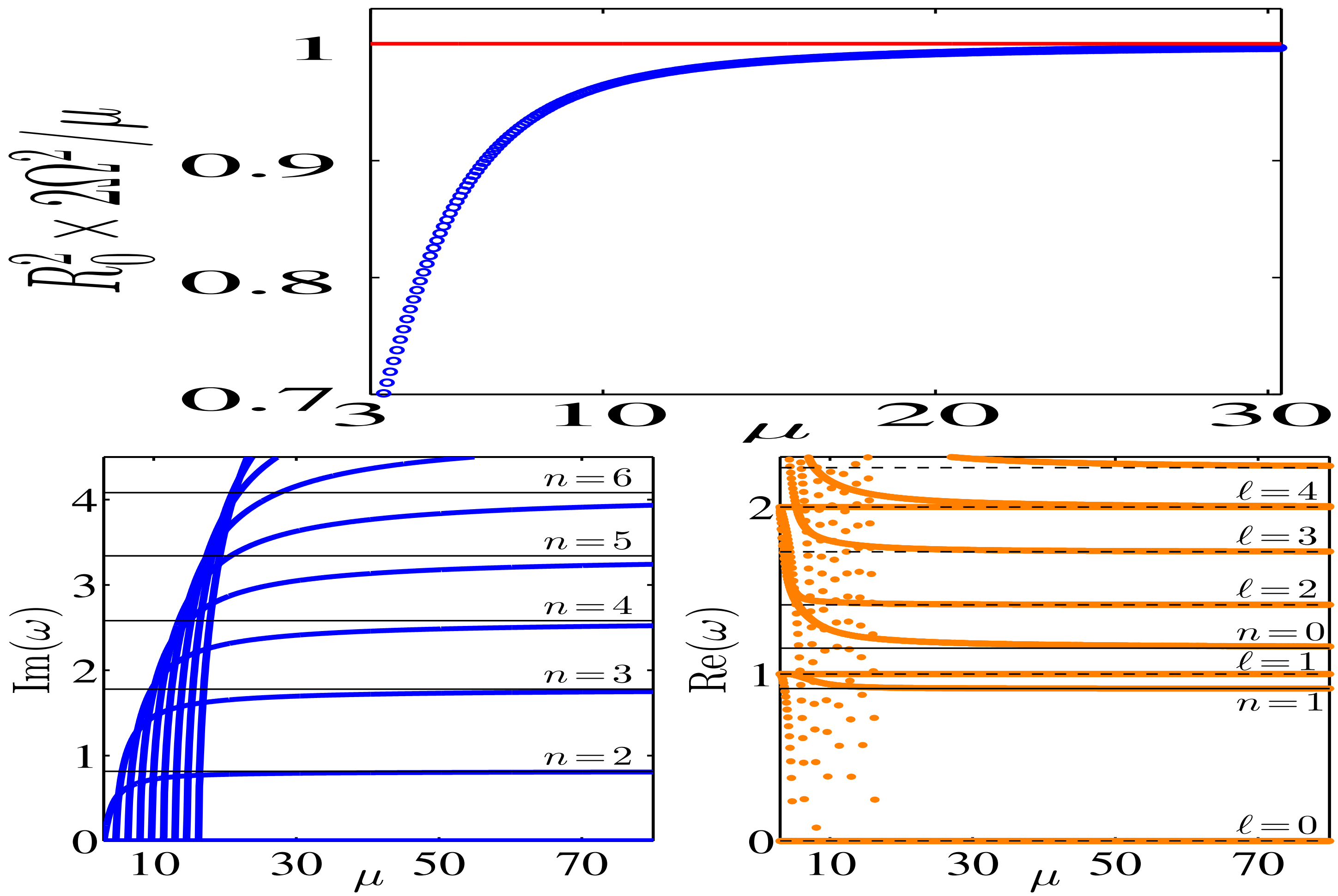
- Identifying the Equilibrium Radius and Linearization Frequencies

$$\frac{(\mu - V(R_0))}{3R_0} = \frac{V'(R_0)}{2}, \quad \text{and} \quad \omega^2 = \frac{V'(R_0)}{2R_0}\left[\frac{5}{3} - n^2 + \frac{R_0V''(R_0)}{V'(R_0)}\right].$$

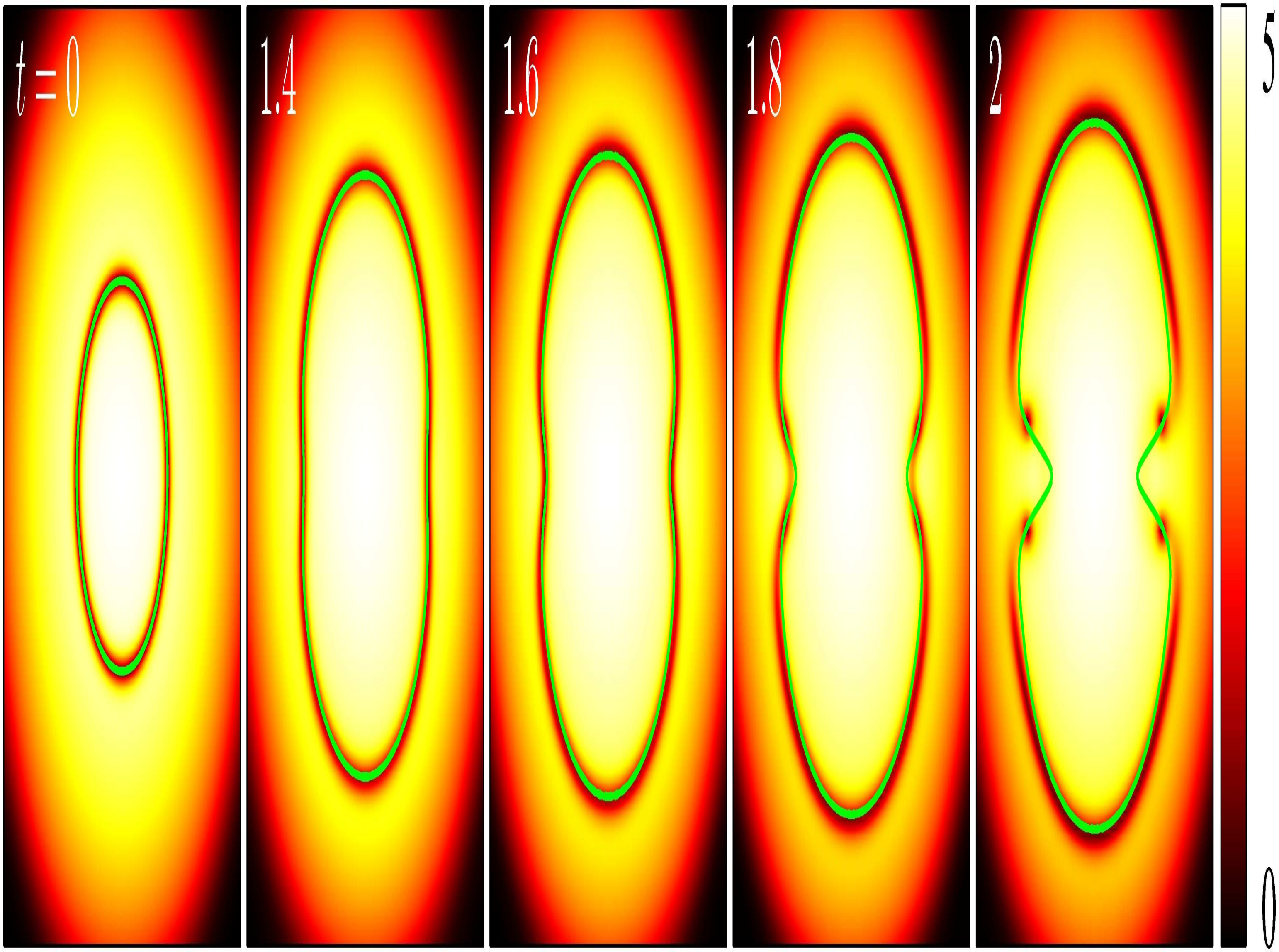
- For the Experimentally Relevant $V(R) = (1/2)\Omega^2R^2$, this yields:

$$R_0^2 = \frac{\mu}{2\,\Omega^2} \quad \text{and} \quad \omega = \pm\left(\frac{1}{2}\left(\frac{8}{3} - n^2\right)\right)^{1/2}\Omega. \qquad (24)$$

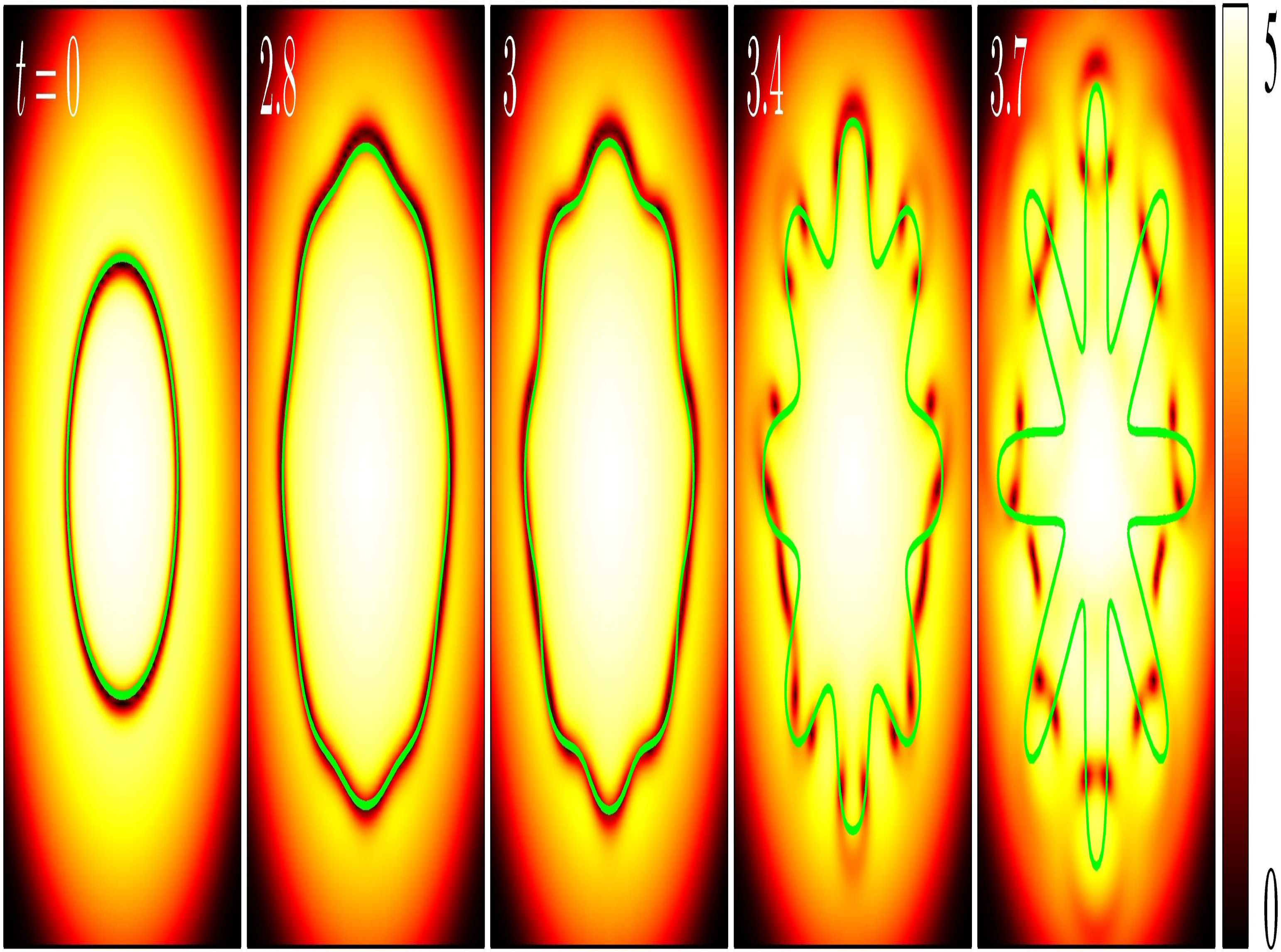
Confirming the Prediction: Existence/Stability of RDS



Confirming the Prediction: Dynamics of RDS



Confirming the Prediction: Dynamics of RDS (Contd.)



3d Extensions: Planar and Spherical Dark Solitons

- For Planar Dark Solitons, the Center Position $\xi = \xi(y,z,t)$ represents an Evolving Surface with:

$$E = \int \left(1 + \frac{1}{2}\xi_y^2 + \frac{1}{2}\xi_z^2\right)\left(\mu - V(\xi) - \xi_t^2\right)^{3/2} dydz.$$

- For Spherical Dark Shells the Radial Position $R = R(\theta,\phi,t)$,

$$E = \frac{4}{3}\int R^2 \left(1 + \frac{R_\theta^2}{2R^2} + \frac{R_\phi^2}{2R^2 \sin^2(\theta)}\right)\left(\mu - R_t^2 - V(R)\right)^{3/2} d\theta d\phi.$$

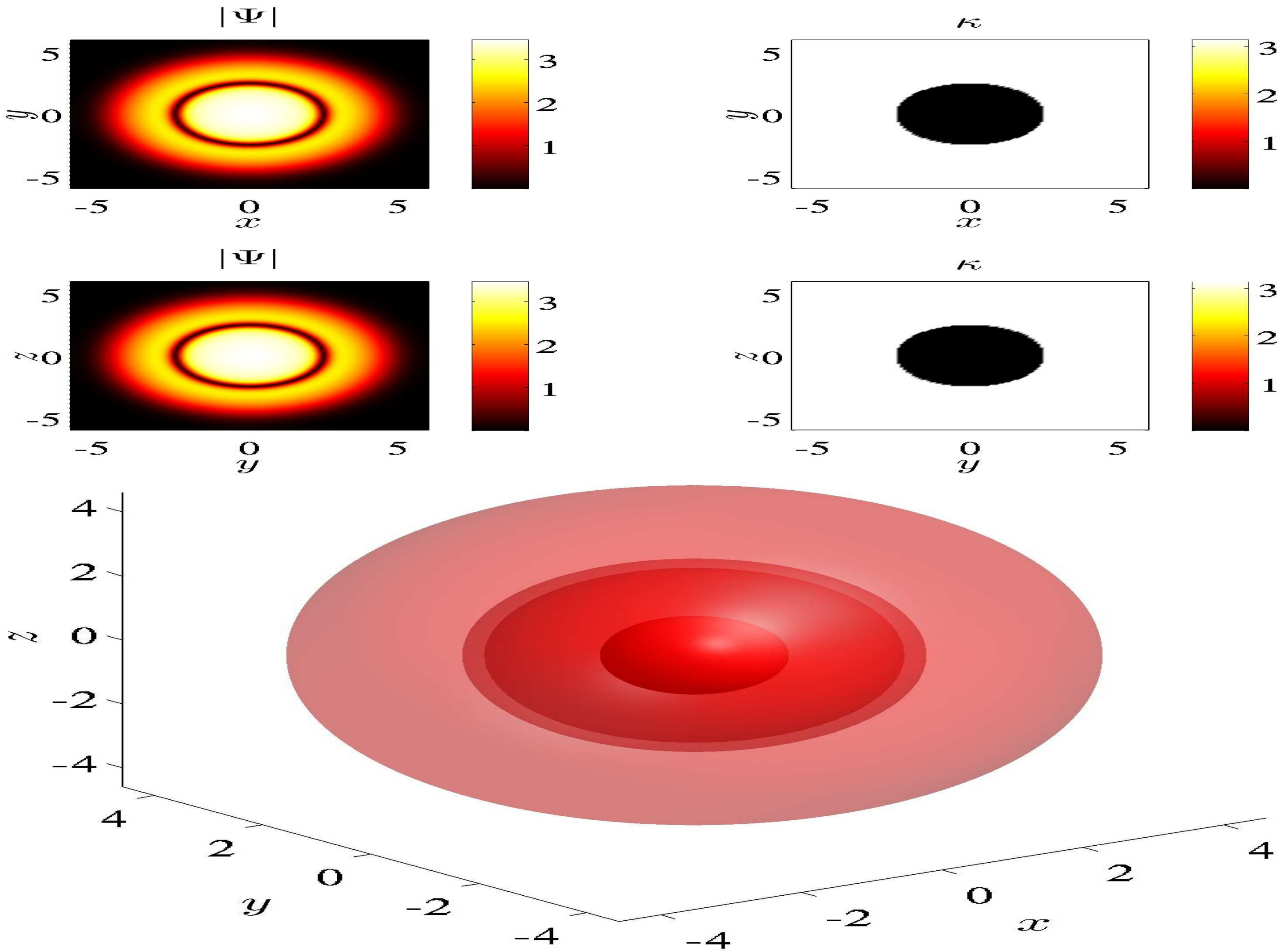
- From this obtain Equilibrium Position and Linearization with $\tilde{A} = \int_0^\pi R_1^2 \sin\theta d\theta$, $\tilde{B} = \int_0^\pi (R_1')^2 \sin\theta d\theta$, and $\tilde{C} = \int_0^\pi R_1^2 \sin\theta d\theta$ $(R_1 = P_n^l(\cos(\theta)))$:

$$\frac{2(\mu - V(R_0))}{3R_0} = \frac{V'(R_0)}{2}, \qquad \frac{\omega^2}{\Omega^2} = \frac{7}{6}\frac{V'(R_0)}{R_0} + \frac{1}{2}V''(R_0) - \frac{V'(R_0)}{4R_0}\left(\frac{\tilde{B}}{\tilde{A}} + n^2\frac{\tilde{C}}{\tilde{A}}\right),$$

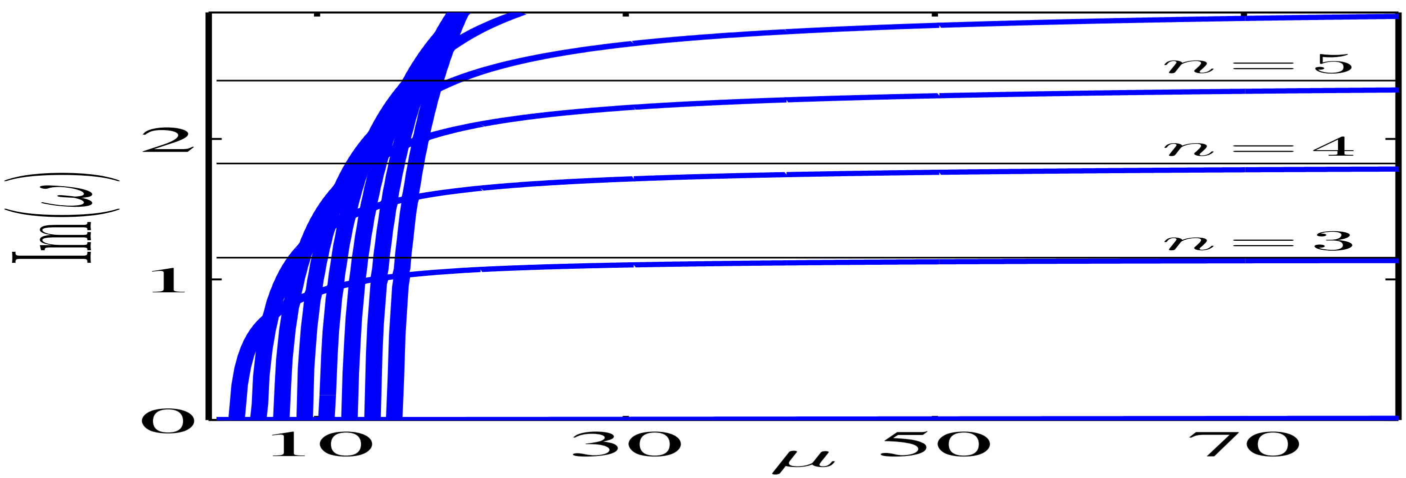
- For the Experimentally Relevant $V(R) = (1/2)\Omega^2 R^2$, this yields:

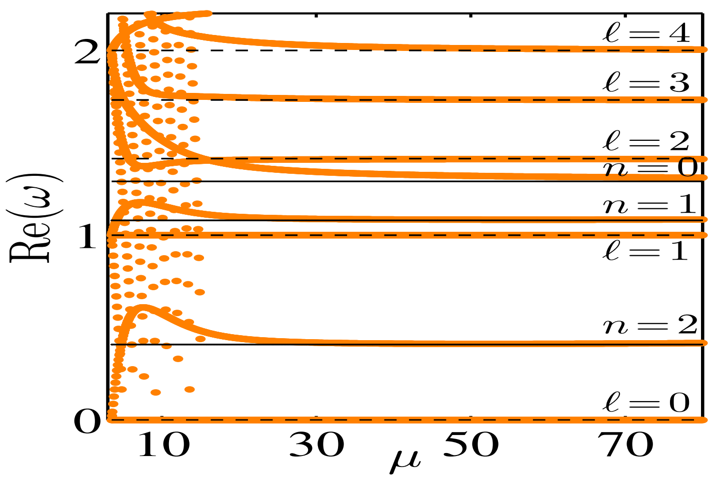
$$\omega^2 = \Omega^2\left(\frac{5}{3} - \frac{1}{4}\left(\frac{\tilde{B}}{\tilde{A}} + n^2\frac{\tilde{C}}{\tilde{A}}\right)\right). \quad (25)$$

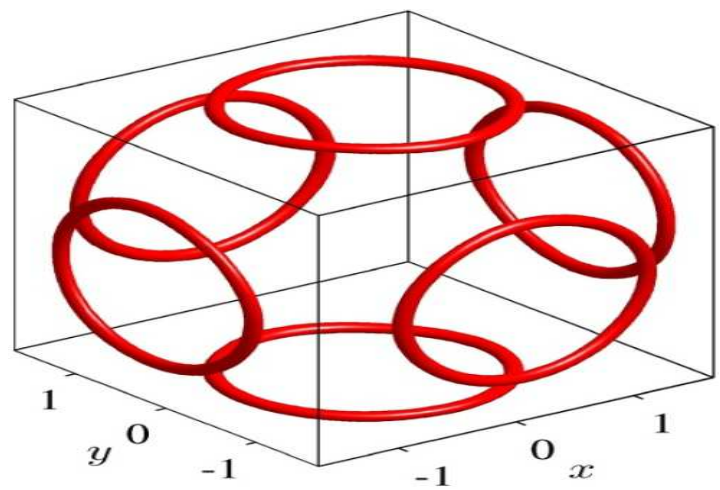
Spherical Dark Shell Solitons: Existence



Spectral Comparison & Dynamics







Multi-Component Extension: Dark-Bright Solitons

- Consider the Manakov Model:

$$
\begin{aligned}
iu_t &= -\frac{1}{2}u_{xx} + \left[V_d + |u|^2 + |v|^2 - \mu_d\right] u,\\
iv_t &= -\frac{1}{2}v_{xx} + \left[V_b + |u|^2 + |v|^2 - \mu_b\right] v.
\end{aligned}
\tag{26}
$$

- The Dark-Bright Solitons are Exact Solutions that read:

$$
u = \sqrt{\mu_d}(\cos(\alpha)\tanh(\nu(x-\xi)) + i\sin(\alpha)), \tag{27}
$$

$$
v = \sqrt{N_b\nu/2}\,\mathrm{sech}(\nu(x-\xi))e^{-i\mu_b t}e^{i\dot{\xi}x}, \tag{28}
$$

- Using: $\mathcal{A}=\mathcal{A}(x)=(\mu_d + N_b^2/16 - V_d(x))^{1/2}$, the DB Free Energy reads:

$$
G_{\mathrm{DB,1D}} = \frac{4}{3}\mathcal{A}^3 - 2\dot{\xi}^2\mathcal{A} + N_b\left(V_b - \frac{1}{2}V_d\right),
$$

- Adding $G_y = \frac{1}{2}\int\left(|u_y|^2 + |v_y|^2\right)dx$, yields the 2D Free Energy:

$$
G_{\mathrm{DB,2D}} = \int G_{\mathrm{DB,1D}} + \xi_y^2\left(\frac{2}{3}\mathcal{A}^3 - \frac{1}{8}N_b^2\mathcal{A} + \frac{1}{48}N_b^3 - \xi_t^2\frac{8\mu_d + N_b^2 - 8V_d}{8\mathcal{A}}\right)dy,
$$

Dark-Bright Solitons Continued

- For Center Position $\xi = X_0 + \epsilon \cos(k_n y) X_1(t)$, the Near Linear Filament Dynamics reads:

$$X_{1tt} = -\omega_n^2 X_1,$$

with (squared) eigenfrequencies

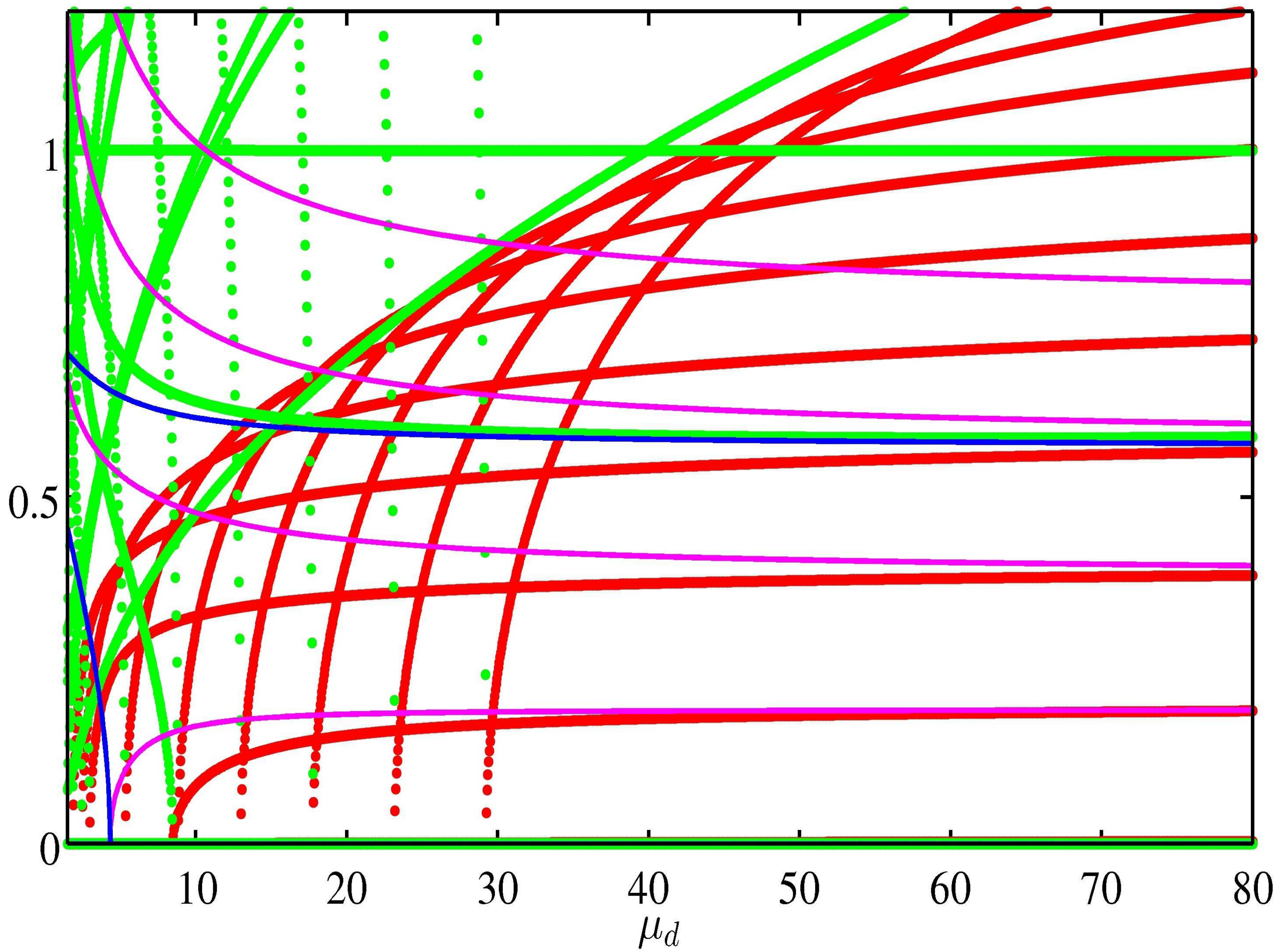
$$\omega_n^2 = \frac{1}{2}V_d'' - \frac{N_b}{4\mathcal{A}_0}\left(V_b'' - \frac{1}{2}V_d''\right) - k_n^2\left(\frac{1}{3}\mathcal{A}_0^2 + \frac{1}{96}\frac{N_b^3}{\mathcal{A}_0} - \frac{1}{16}N_b^2\right), \qquad (29)$$

- For the Experimentally Relevant $V_{d,b}(x) = (1/2)\Omega^2 x^2$, this yields:

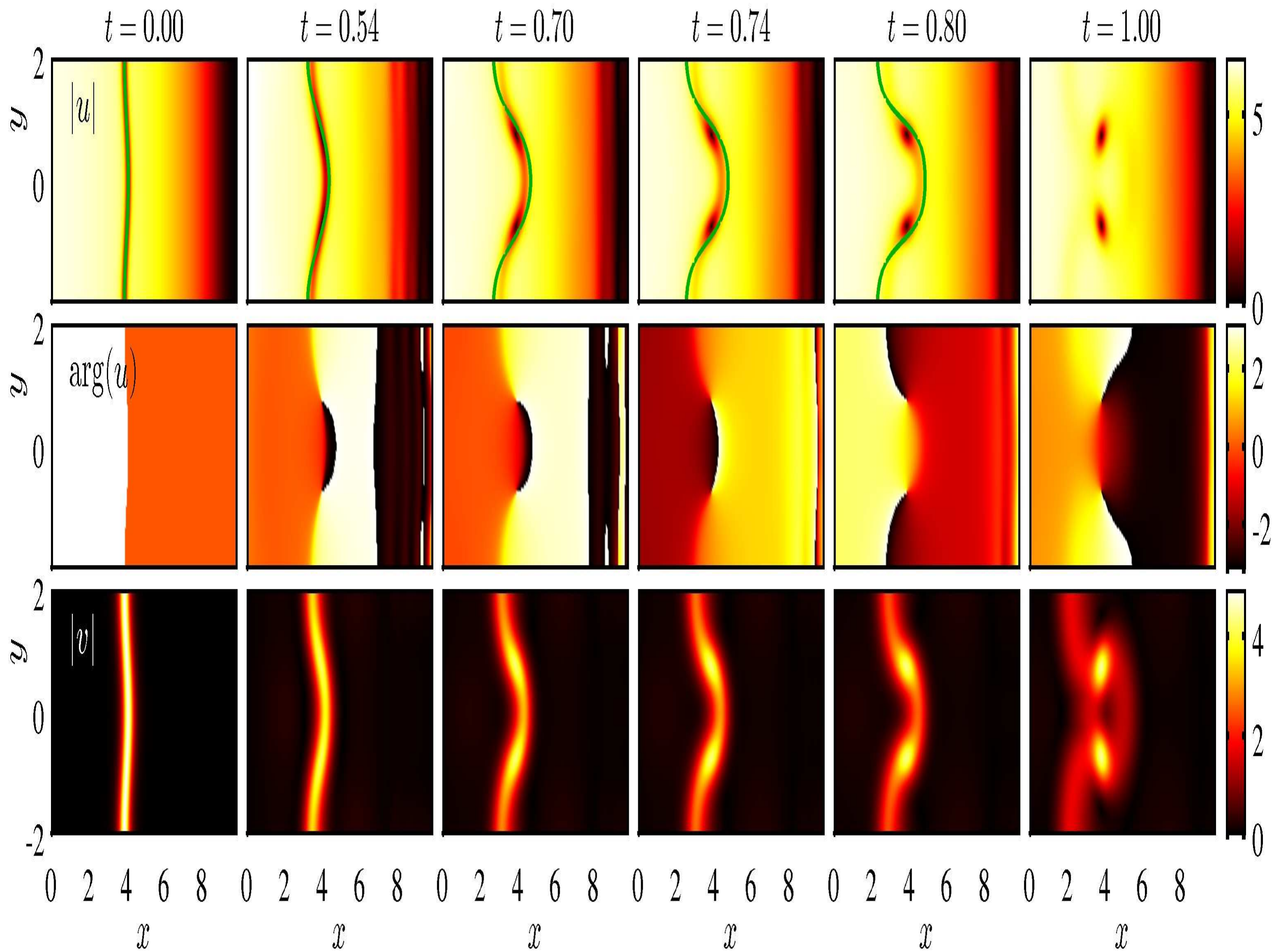
$$\omega_n^2 = \frac{1}{2}\Omega^2 - \frac{N_b}{8\mathcal{A}_0}\Omega^2 - \frac{1}{3}\mu_d k_n^2 - \left(\frac{N_b}{4\mathcal{A}_0} - 1\right)\frac{N_b^2\, k_n^2}{24}. \qquad (30)$$

- This encompasses Dark 1D (1st term), Bright Contribution 1D (2nd term), Dark Transverse Effect (3rd term) and Bright Transverse Effect (4th term).

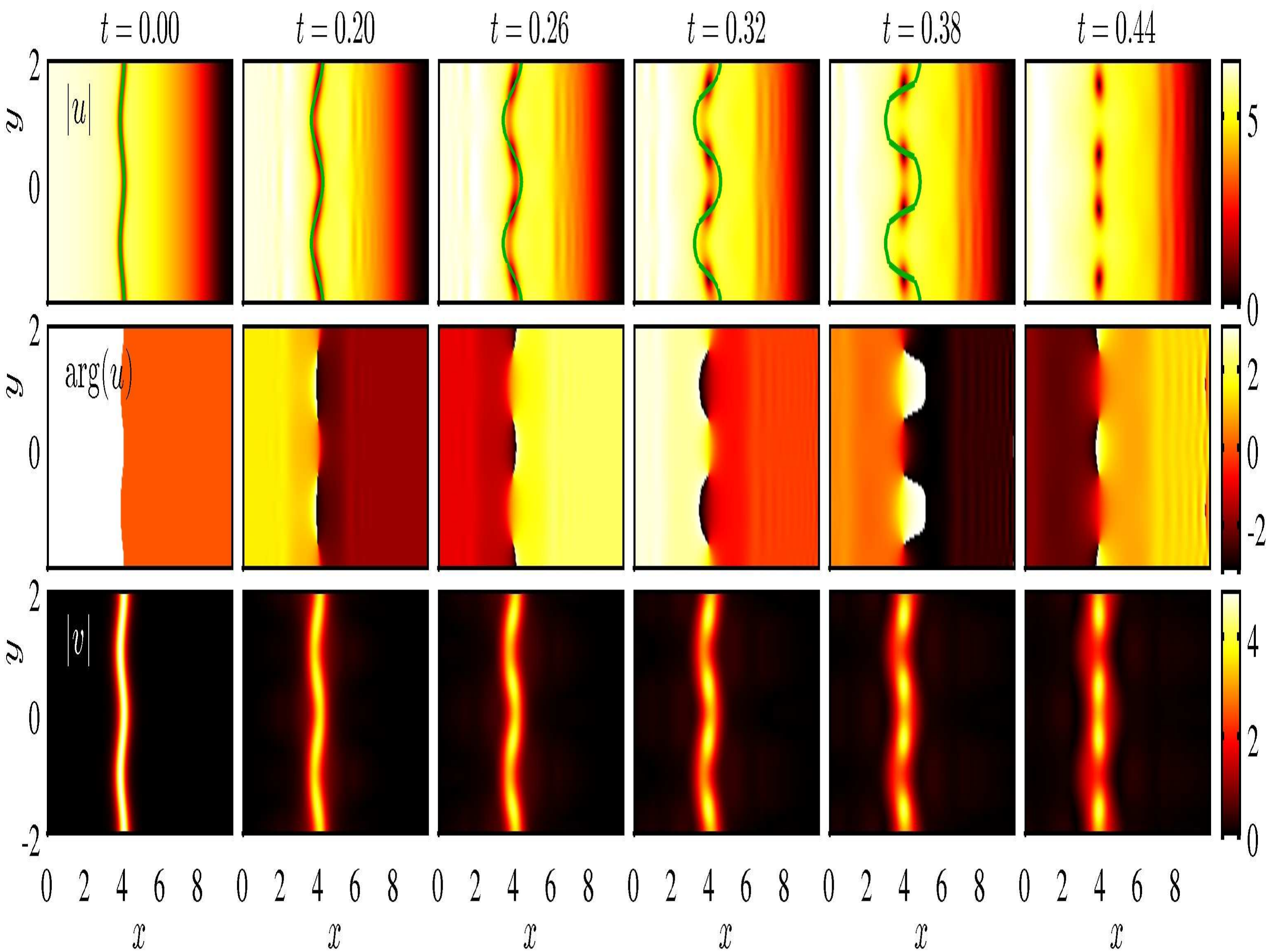
Spectral Comparison

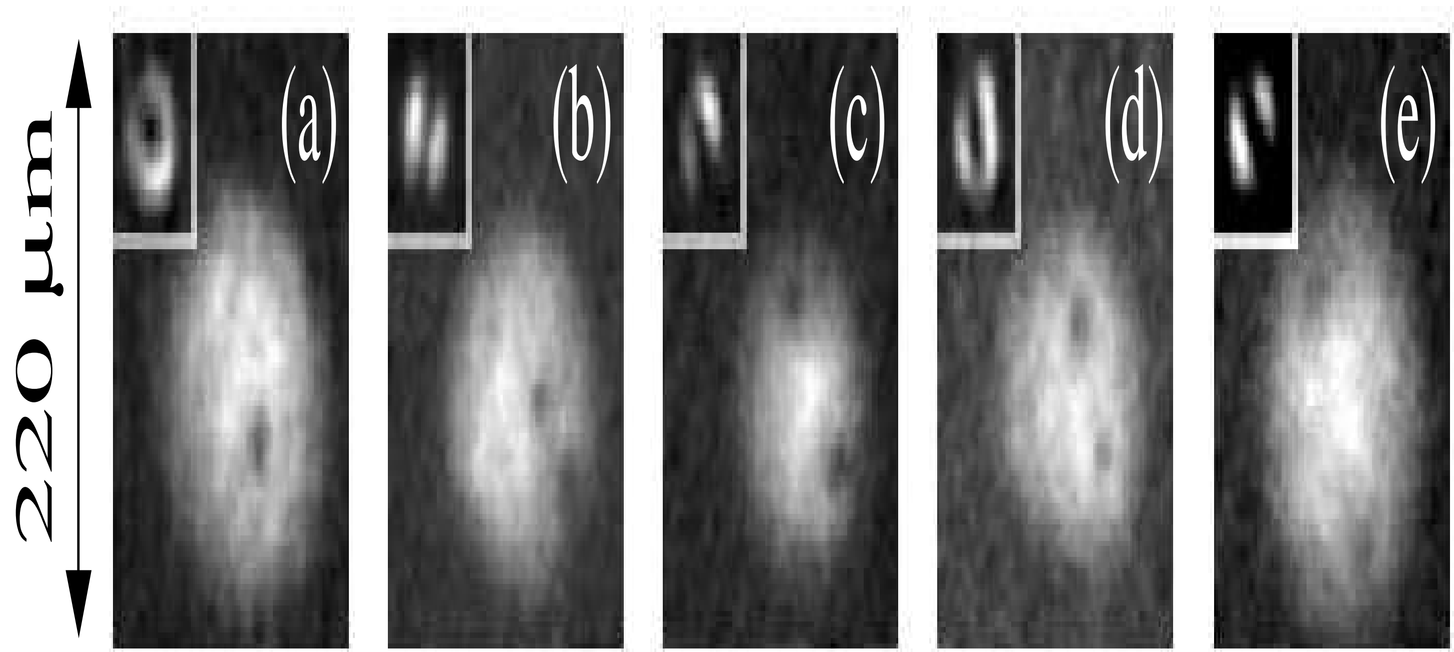


DB Line Transverse Instability

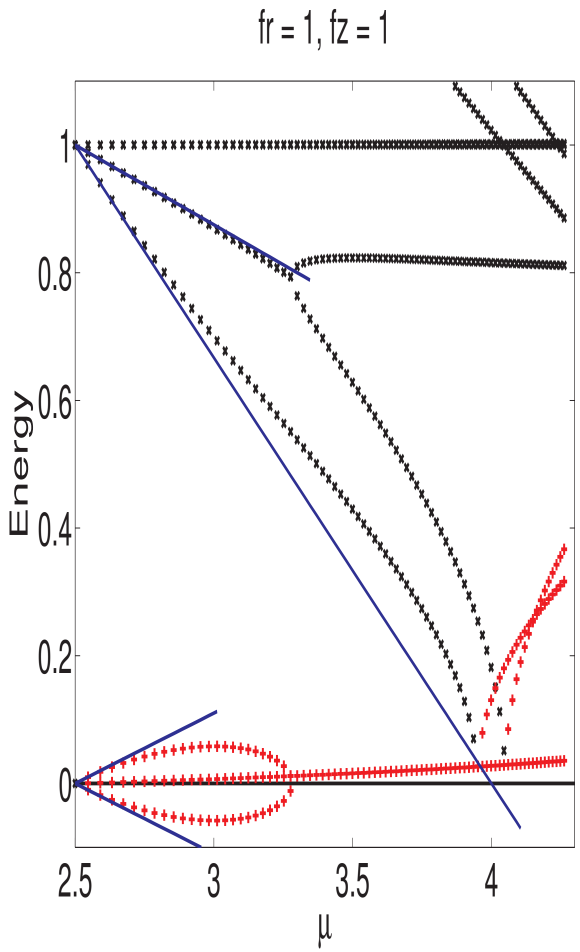


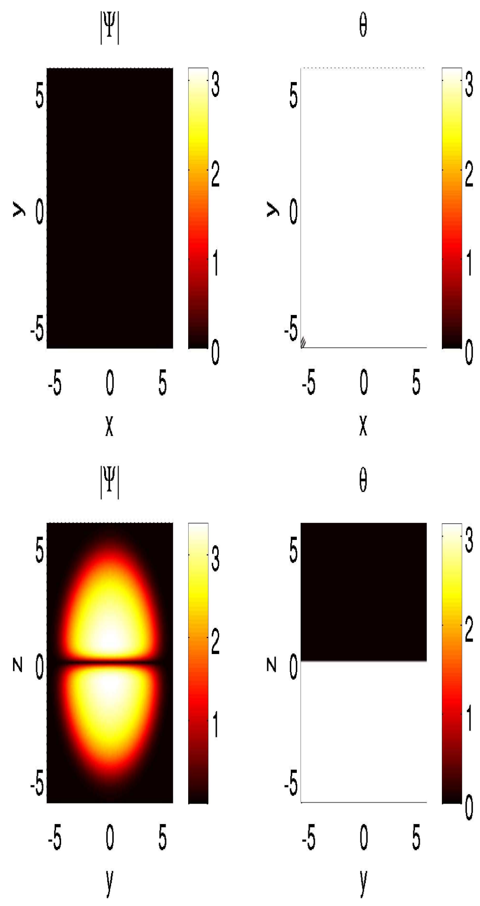
DB Line Transverse Instability (Contd.)

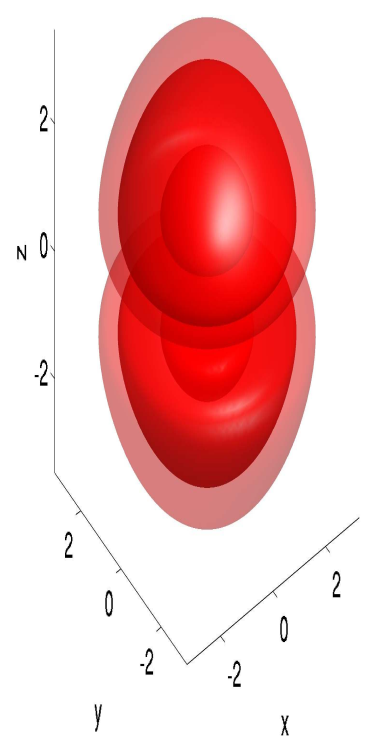



220 μm
(a)
(b)
(c)
(d)
(e)

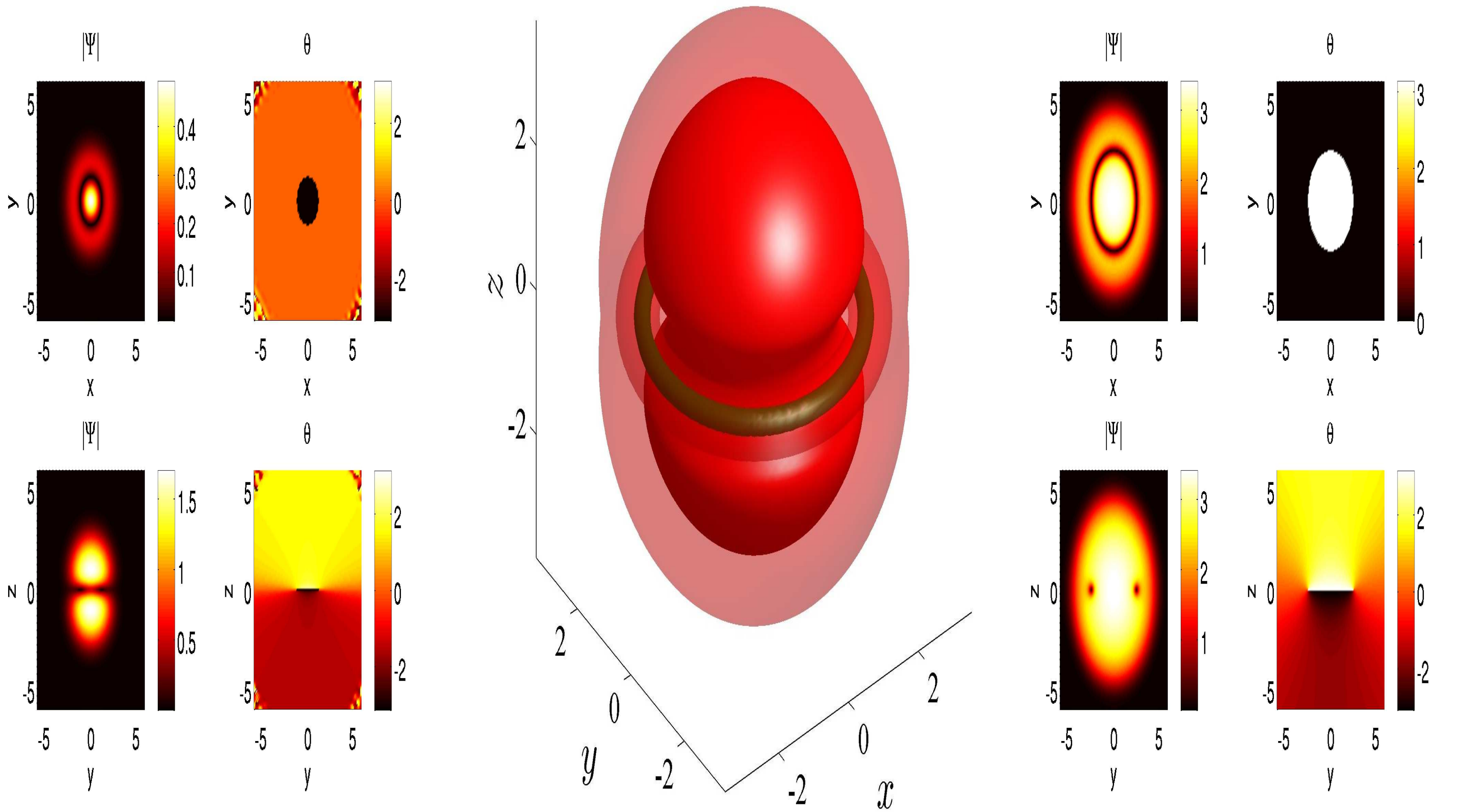
Planar Dark Solitons in 3d (Contd.)







Bifurcating Single Vortex Ring in 3d



Dynamical Formulation for Vortex Ring

- Consider the Lagrangian Formulation for a Vortex Ring (see, Ruban's work: e.g., arXiv:1706.04348 (published in JETP Letters))

$$L = \int F(R,Z)Z_t - \rho(R,Z)\sqrt{R^2 + R_\theta^2 + Z_\theta^2}\,d\theta \tag{31}$$

Here, F is a function such that $F_R = \rho(R,Z)R$ and the Asymptotic (TF) density $\rho = \mu - V(R,Z)$.

- Then, the PDEs describing the R- and Z-motion of the VR read (with $A = \sqrt{R^2 + R_\theta^2 + Z_\theta^2}$ (the Cylindrical Arclength):

$$\rho R R_t = -\rho_z A + \frac{\partial}{\partial\theta}\left(\frac{\rho Z_\theta}{A}\right) \tag{32}$$

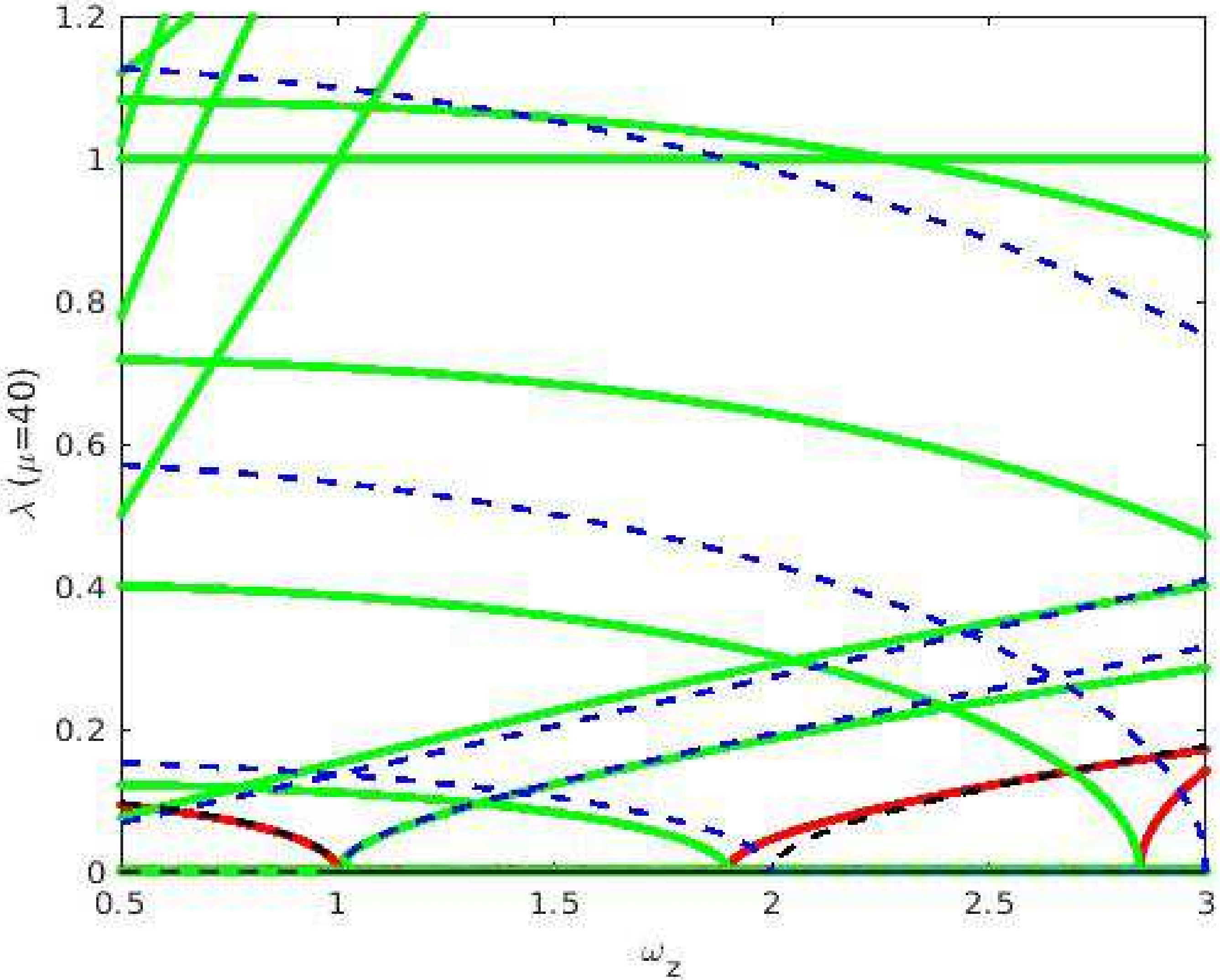
$$\rho R Z_t = \rho_R A + \frac{\rho}{A} R - \frac{\partial}{\partial\theta}\left(\frac{\rho Z_\theta}{A}\right) \tag{33}$$

- From this obtain Equilibrium with $Z = 0$ and $R = R_0 = (2\mu)/(3\Omega_R^2)$ and Linearizing with $R = R_0 + \sum \epsilon R_m \cos(m\theta)$ and $Z = \sum \epsilon Z_m \cos(m\theta)$, we obtain the Frequencies:

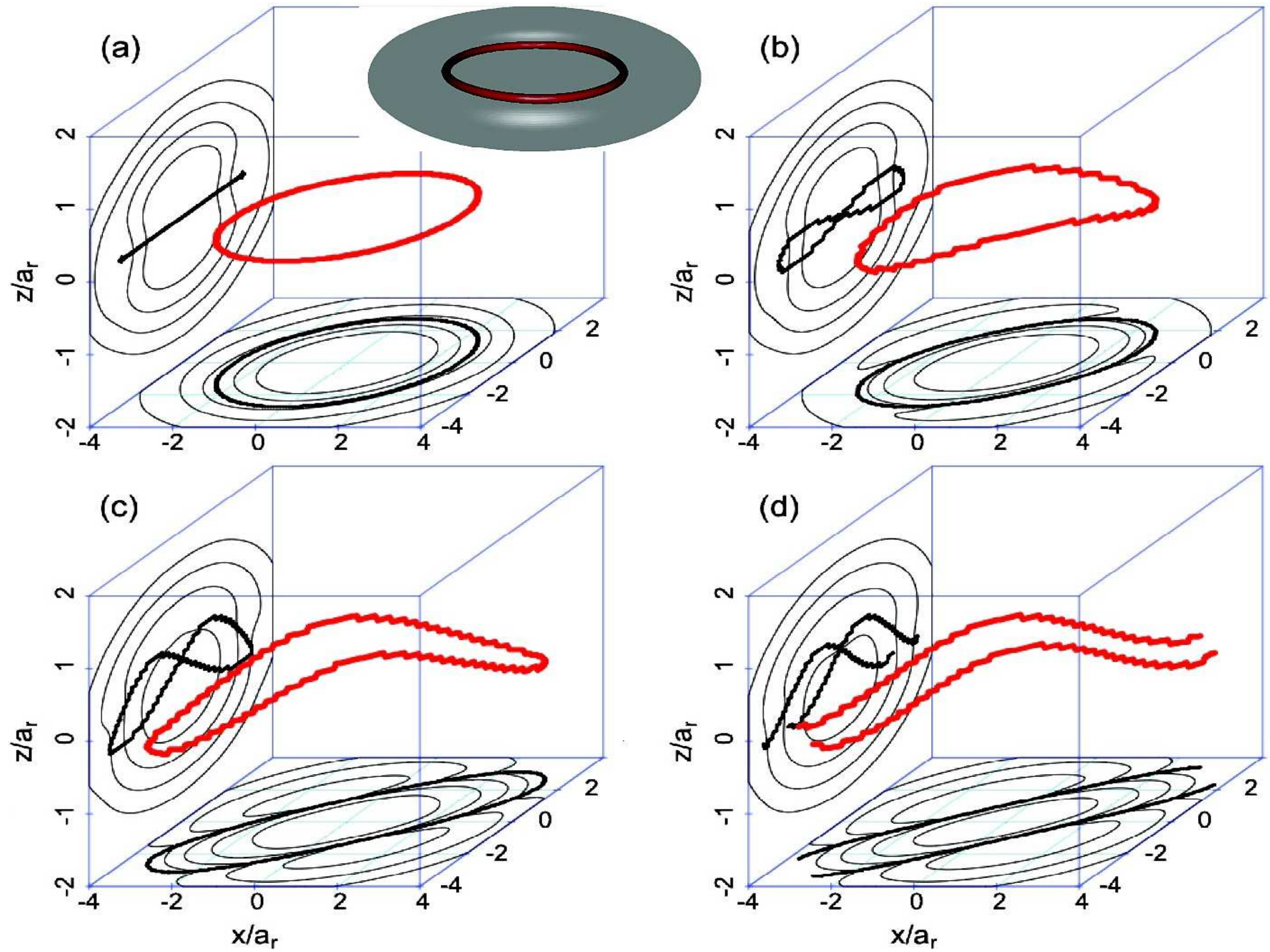
$$\omega = \frac{3}{R_\perp^2}\left((m^2 - \tilde{\lambda}^2)(m^2 - 3)\right)^{1/2} \tag{34}$$

- Conclusion: Rings for $1 < \tilde{\lambda} = \Omega_z/\Omega_R < 2$, Stable, Otherwise Unstable.

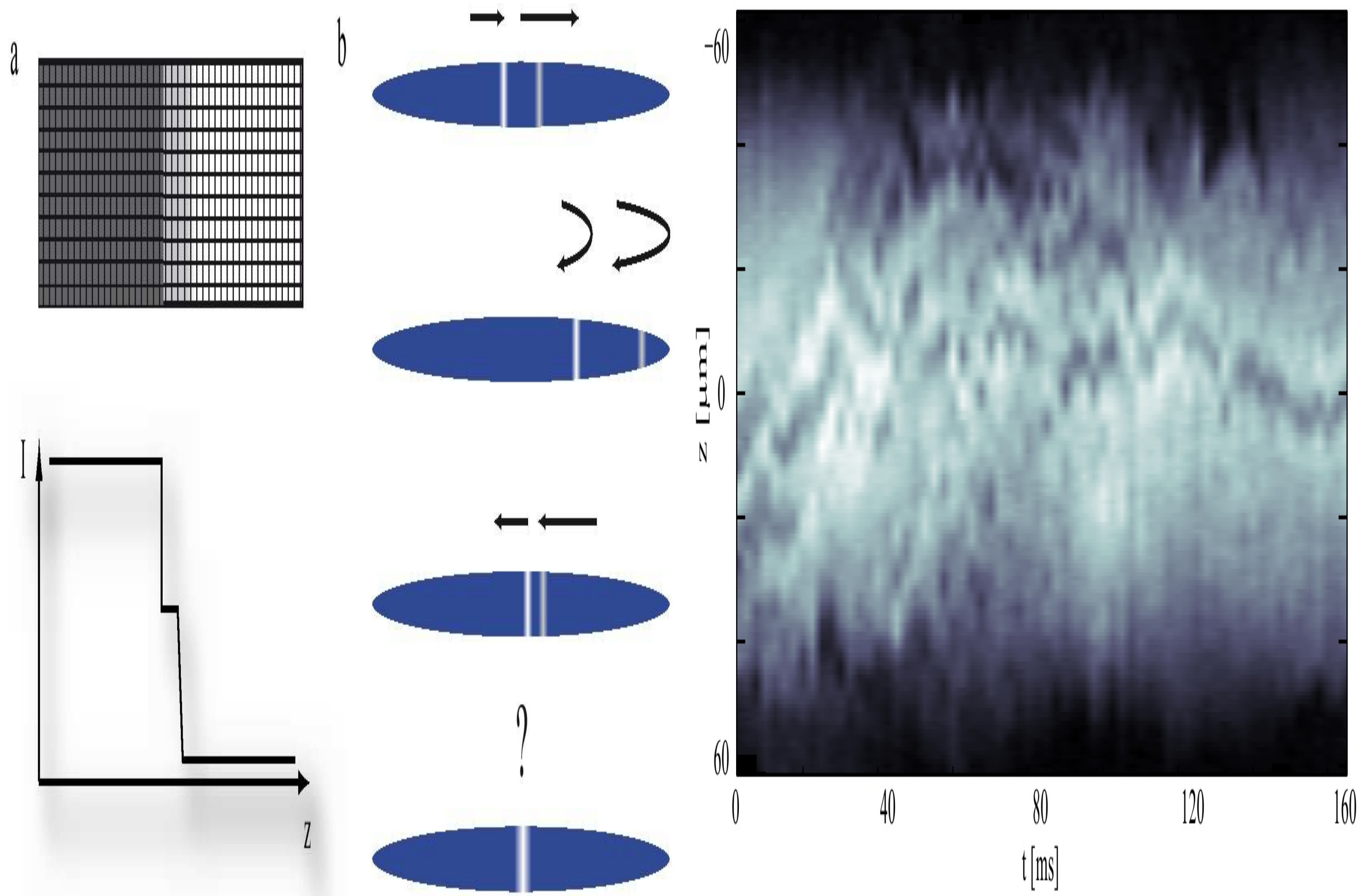
Spectral Comparison



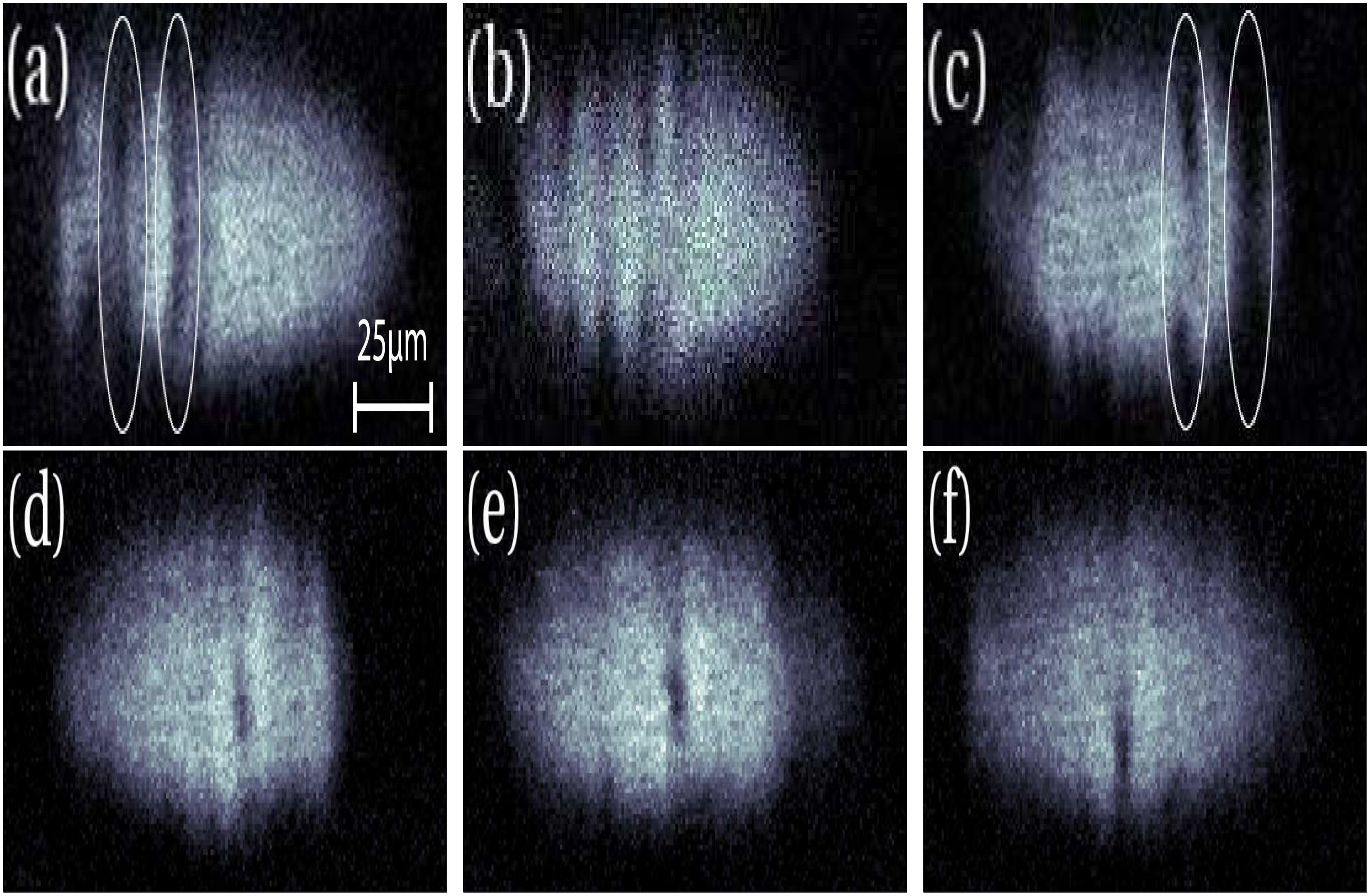
VR Instability Dynamical Evolution



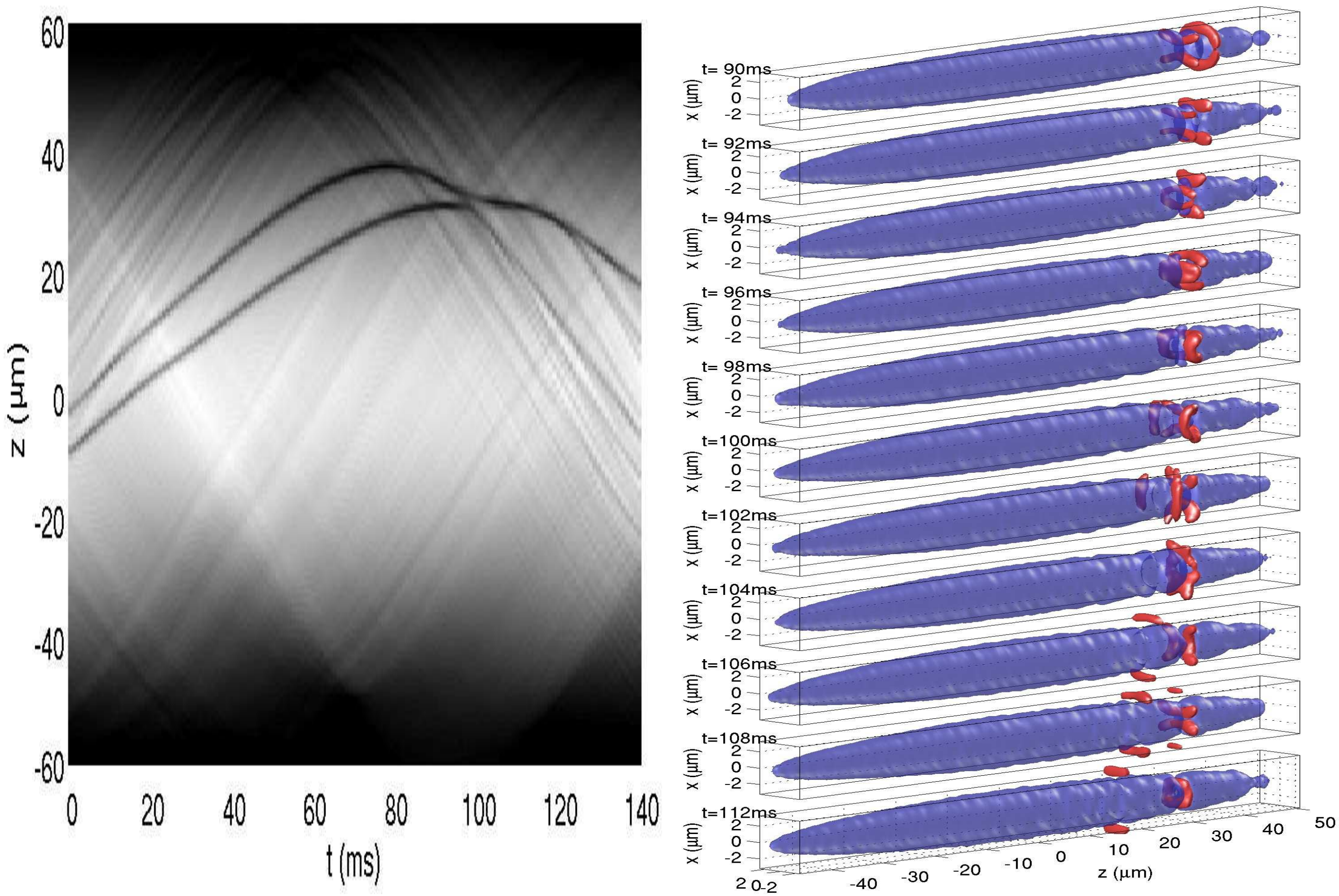
Usefulness for Understanding Experiments: VL/VR Collisions



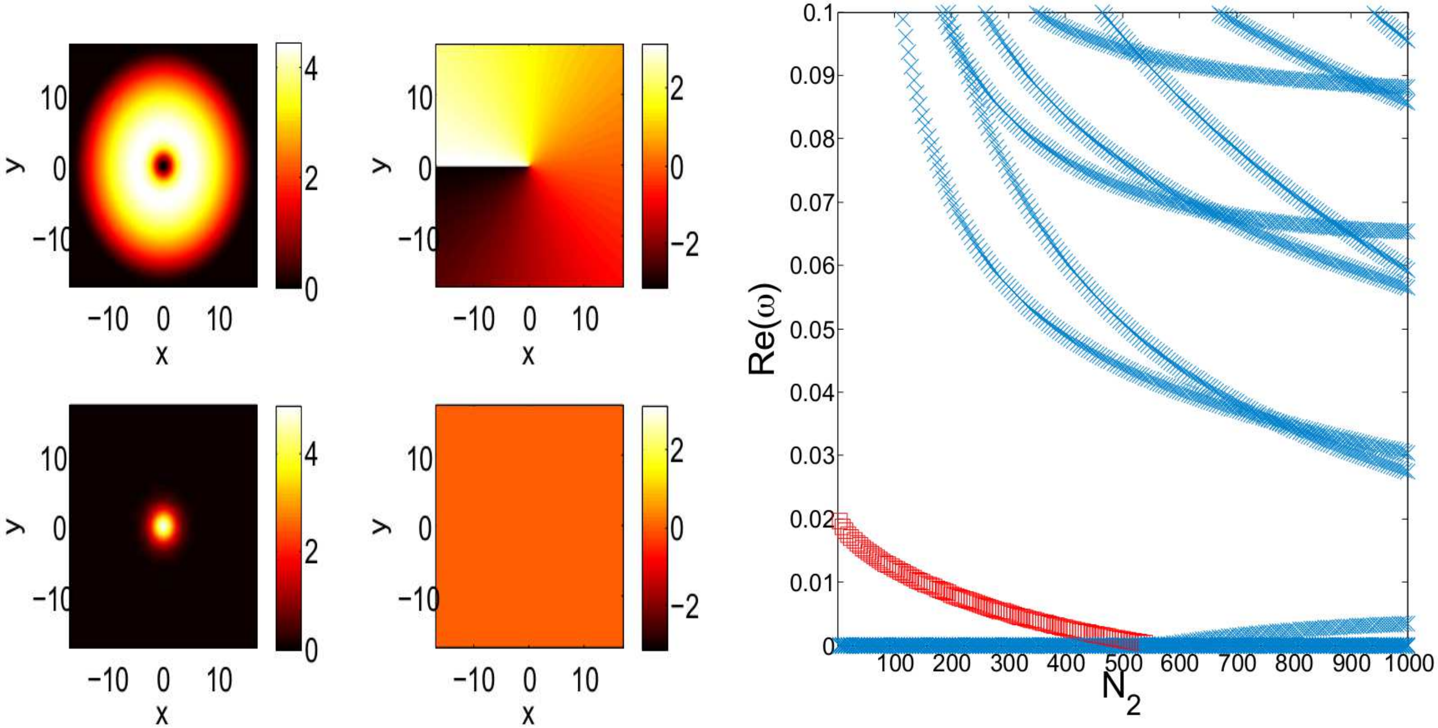
Details of VL/VR Collision Experiments



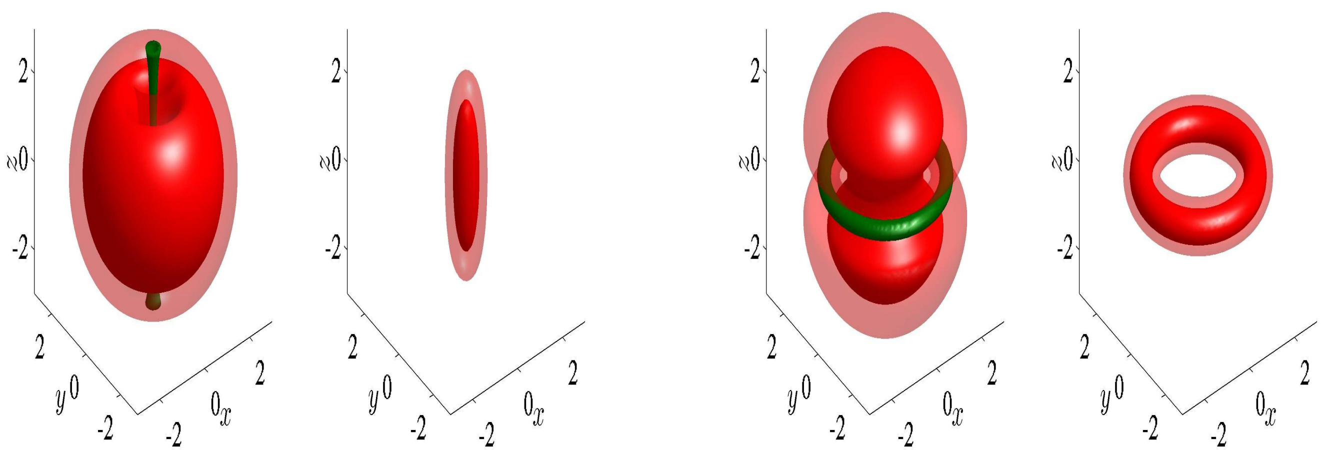
A Theoretical Understanding of VL/VR Collision Experiments



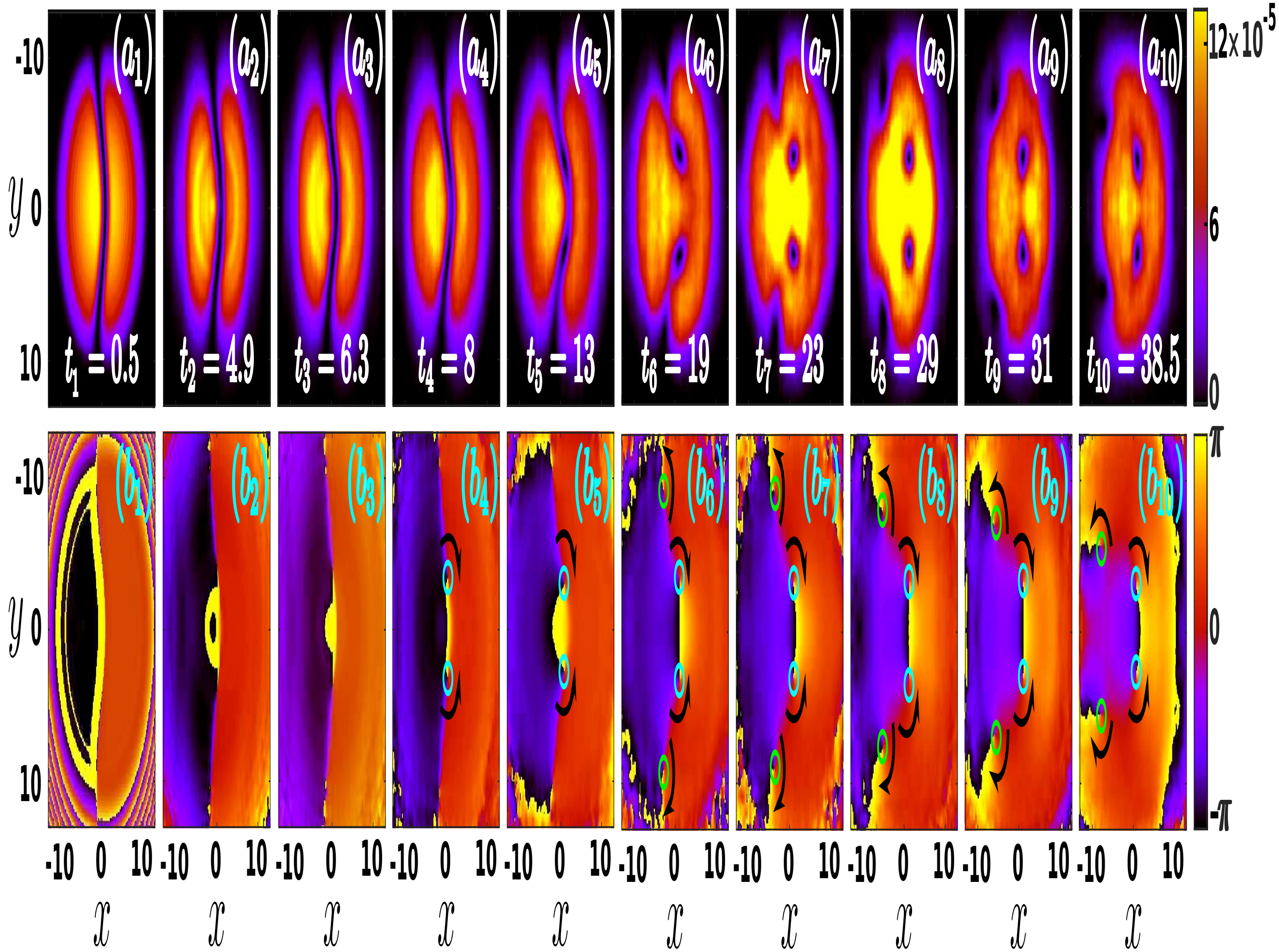
2D Extension: A Single Vortex-Bright Soliton



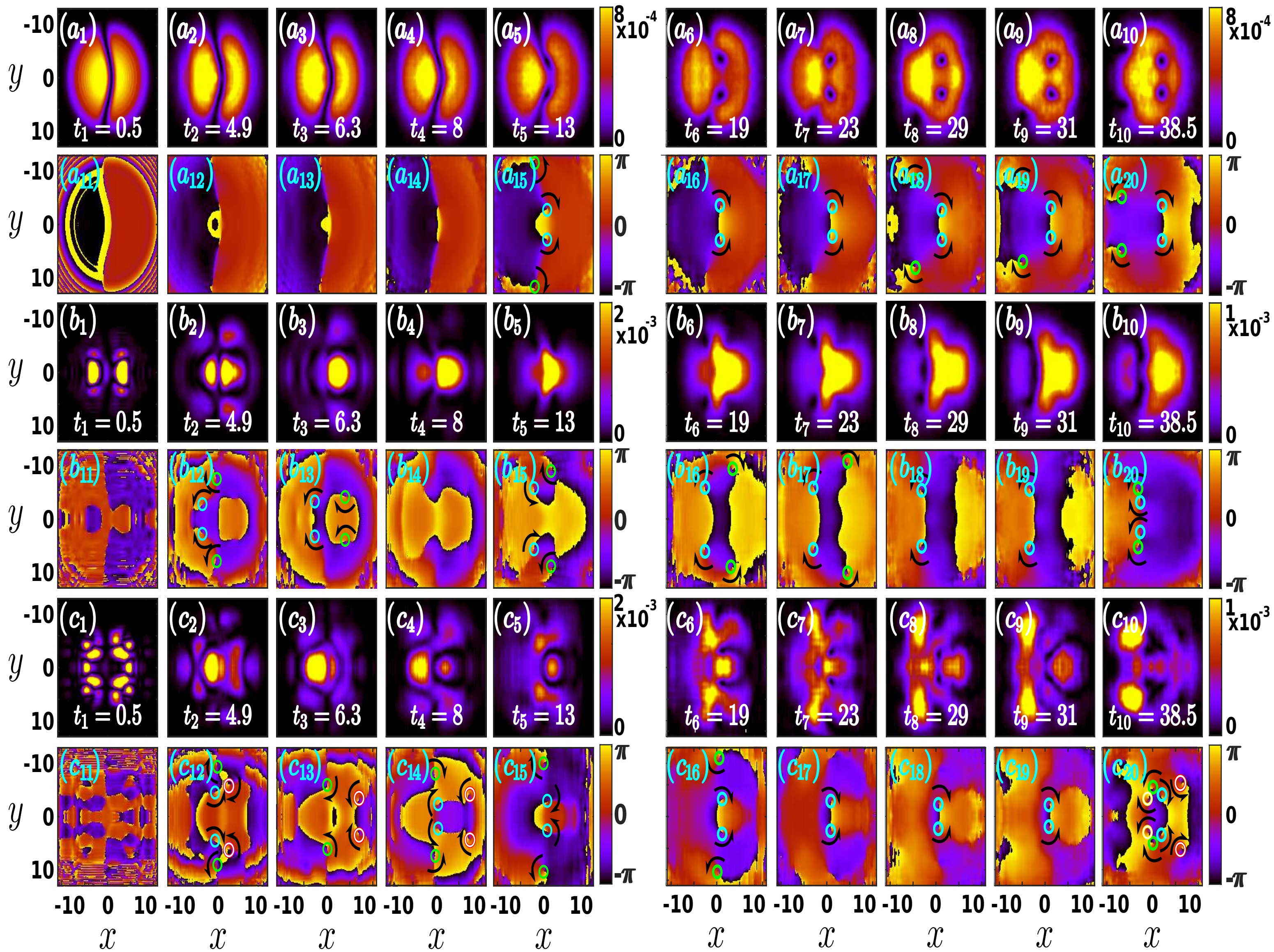
3D Extensions: Vortex Line-Bright and Vortex-Ring Bright



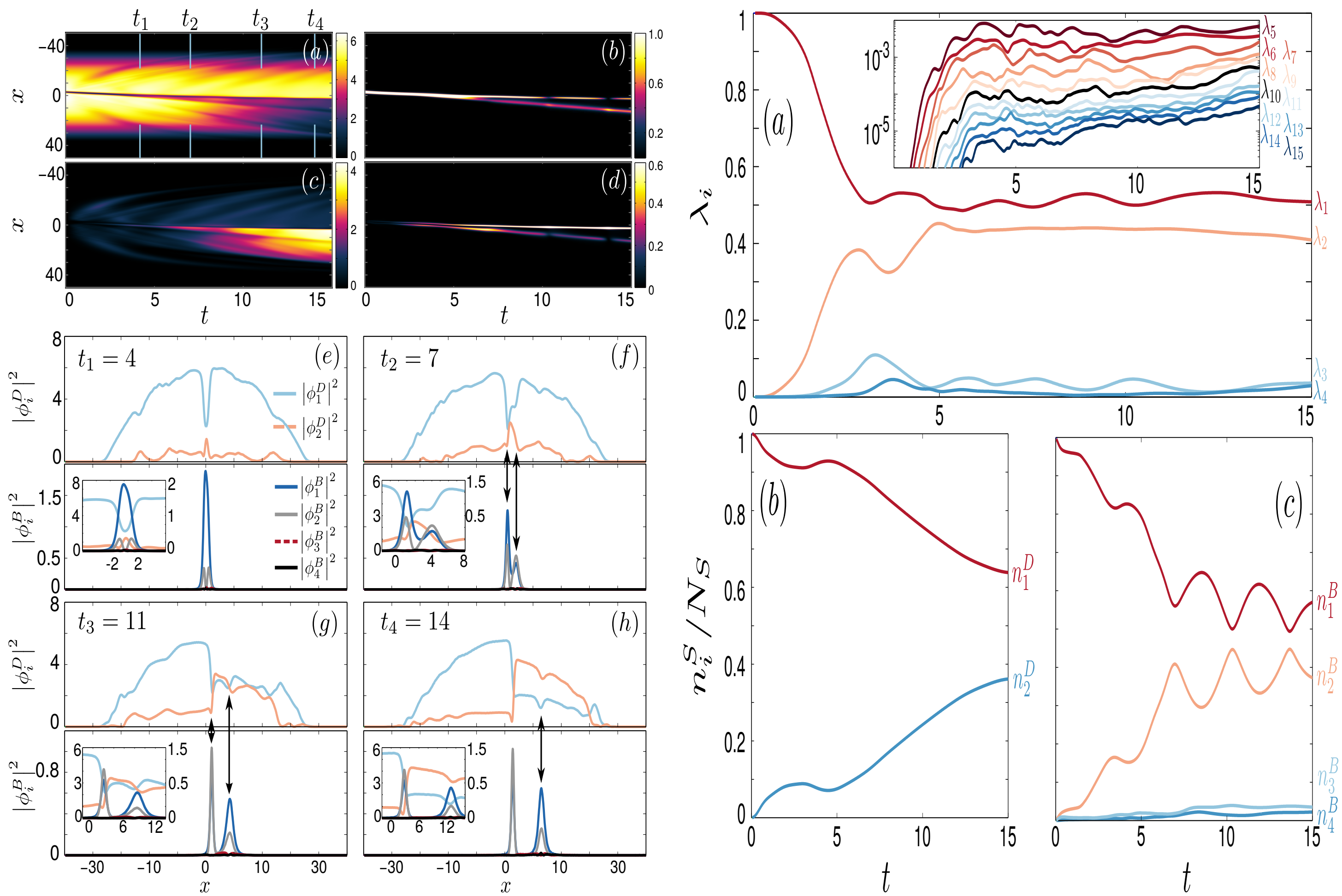
Beyond Mean-Field: Transverse Instability of Bent Dark Solitons



Beyond Mean-Field: Bent Dark Solitons (Contd.)



Beyond Mean-Field: Dark-Bright Solitons



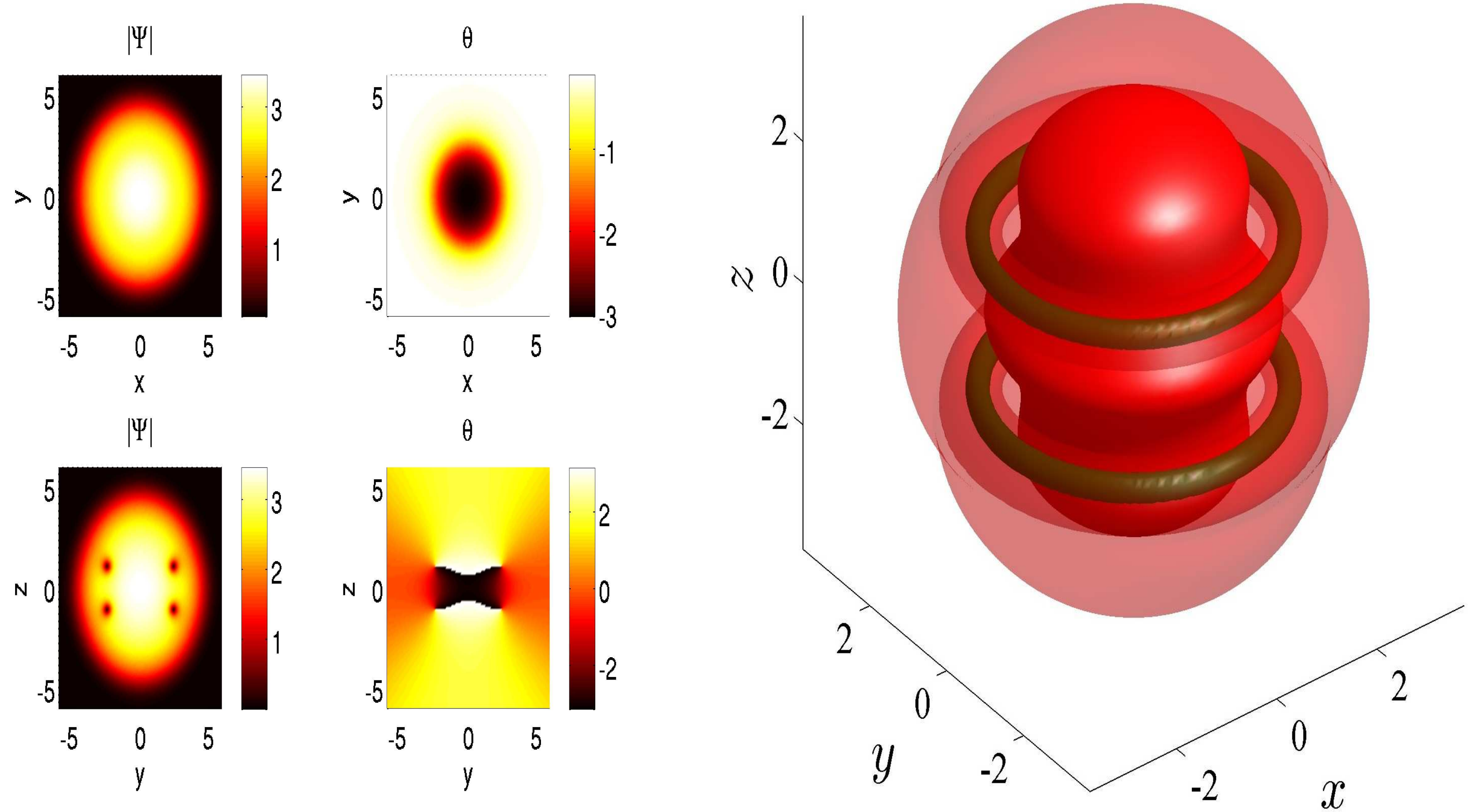
Summary of Results

- Gave Physical Motivation for the study of Dark & Dark-Bright Filament Coherent Structures, especially in Higher Dimensional Settings.
- Unveiled Transverse Instabilities by means of Adiabatic Invariant Approach in 2D, and 3D. Used it to obtain Steady States, Stability and Dynamics.
- Considered Extensions to Ring Dark Solitons and to Planar, as well as Shell Dark Solitons.
- Considered Generalizations to Multi-components exploring the case of Dark-Bright Solitons and their Higher-dimensional Analogues.
- Also Examined Variations towards Vortex Ring Filaments and their Existence/Stability/Dynamics.
- Demonstrated Practical Usefulness of these considerations in Explaining Observations of Higher Dimensional Experiments.

Present/Future Challenges

- One can examine Multi-dimensional Case of Interacting Dark Solitons
- One can consider Connection of Collapse in the context of Filament Equations with Formation Time of Vortical Patterns.
- It is relevant to Examine Dynamics in Intrinsic Variables such as Arclength-Normal.
- Determine Filamentary Description of Vortex Lines and their Kelvin Waves.
- Describe Multi-Component, Multi-Dimensional Structures as Filaments.
- Can Something be explored in the Quantum, Many-Body Case ?

Bifurcating Double Vortex Ring in 3d



Bifurcating Triple Vortex Ring in 3d