# M-Theory as a Dynamical System

Georgios Linardopoulos

Institute of Nuclear & Particle Physics
National Center for Scientific Research "Demokritos"

Dynamical Systems & Complexity Summer School & Conference
Athens, July 10th 2018

based on Phys.Rev. **D97** (2018) 126019 [arXiv:1712.06544] and Phys.Lett. **B773** (2017) 265 [arXiv:1707.02878], with Minos Axenides and Emmanuel Floratos

# Outline

# What is M-theory?

# Quantum field theory

- The $SU(3) \times SU(2) \times U(1)$ Standard Model of elementary particles is the crowning achievement of the effort to unify the Special theory of Relativity with Quantum Mechanics...

# Quantum field theory

- The $SU(3) \times SU(2) \times U(1)$ Standard Model of elementary particles is the crowning achievement of the effort to unify the Special theory of Relativity with Quantum Mechanics: Quantum Field Theory...

# Quantum field theory

- The $SU(3) \times SU(2) \times U(1)$ Standard Model of elementary particles is the crowning achievement of the effort to unify the Special theory of Relativity with Quantum Mechanics: Quantum Field Theory...
- The SM provides a unifying framework for 3 of the fundamental forces (strong, weak, electromagnetic)...

# Quantum field theory

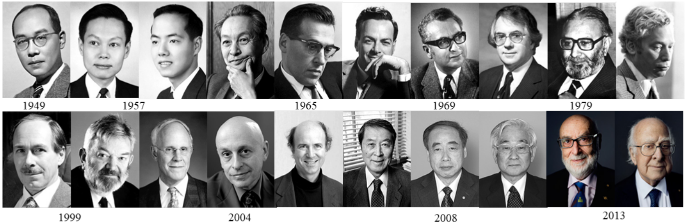
- The $SU(3) \times SU(2) \times U(1)$ Standard Model of elementary particles is the crowning achievement of the effort to unify the Special theory of Relativity with Quantum Mechanics: Quantum Field Theory...
- The SM provides a unifying framework for 3 of the fundamental forces (strong, weak, electromagnetic)...
- It has been spectacularly confirmed...

# Quantum field theory

- The $SU(3) \times SU(2) \times U(1)$ Standard Model of elementary particles is the crowning achievement of the effort to unify the Special theory of Relativity with Quantum Mechanics: Quantum Field Theory...
- The SM provides a unifying framework for 3 of the fundamental forces (strong, weak, electromagnetic)...
- It has been spectacularly confirmed... the latest discovery being the BEH boson from LHC in 2012...

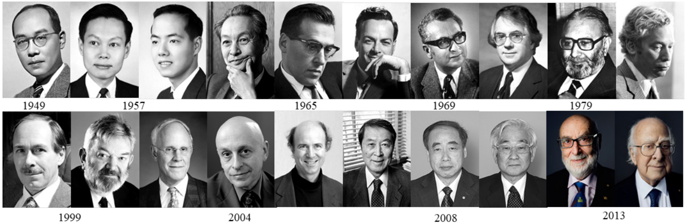
# Quantum field theory

- The $SU(3) \times SU(2) \times U(1)$ Standard Model of elementary particles is the crowning achievement of the effort to unify the Special theory of Relativity with Quantum Mechanics: Quantum Field Theory...
- The SM provides a unifying framework for 3 of the fundamental forces (strong, weak, electromagnetic)...
- It has been spectacularly confirmed... the latest discovery being the BEH boson from LHC in 2012...
- More than 15 Nobel prizes since 1950...



# Quantum field theory

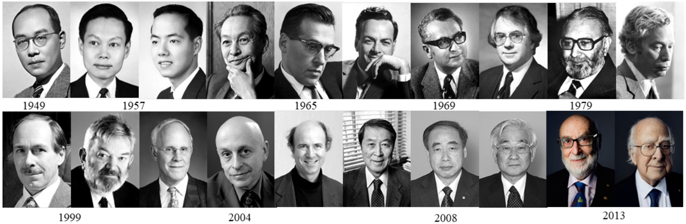
- The $SU(3) \times SU(2) \times U(1)$ Standard Model of elementary particles is the crowning achievement of the effort to unify the Special theory of Relativity with Quantum Mechanics: Quantum Field Theory...
- The SM provides a unifying framework for 3 of the fundamental forces (strong, weak, electromagnetic)...
- It has been spectacularly confirmed... the latest discovery being the BEH boson from LHC in 2012...
- More than 15 Nobel prizes since 1950...




- The model has 19 free parameters (10 masses, 3 gauge couplings, 5 angles, Higgs vev)...

# Quantum field theory

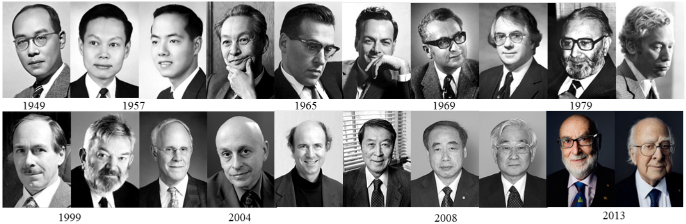
- The $SU(3) \times SU(2) \times U(1)$ Standard Model of elementary particles is the crowning achievement of the effort to unify the Special theory of Relativity with Quantum Mechanics: Quantum Field Theory...
- The SM provides a unifying framework for 3 of the fundamental forces (strong, weak, electromagnetic)...
- It has been spectacularly confirmed... the latest discovery being the BEH boson from LHC in 2012...
- More than 15 Nobel prizes since 1950...




- The model has 19 free parameters (10 masses, 3 gauge couplings, 5 angles, Higgs vev)... Particles: quarks & antiquarks, leptons & antileptons, gauge & Higgs bosons...

# Quantum field theory

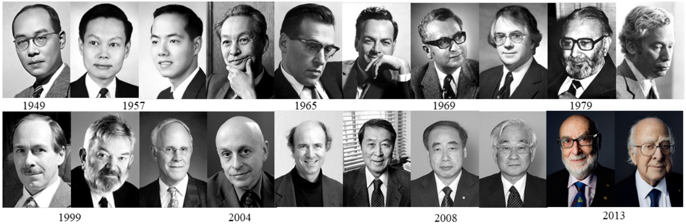
- The $SU(3) \times SU(2) \times U(1)$ Standard Model of elementary particles is the crowning achievement of the effort to unify the Special theory of Relativity with Quantum Mechanics: Quantum Field Theory...
- The SM provides a unifying framework for 3 of the fundamental forces (strong, weak, electromagnetic)...
- It has been spectacularly confirmed... the latest discovery being the BEH boson from LHC in 2012...
- More than 15 Nobel prizes since 1950...




- The model has 19 free parameters (10 masses, 3 gauge couplings, 5 angles, Higgs vev)... Particles: quarks & antiquarks, leptons & antileptons, gauge & Higgs bosons...
- What determines the free parameters?

# Quantum field theory

- The $SU(3) \times SU(2) \times U(1)$ Standard Model of elementary particles is the crowning achievement of the effort to unify the Special theory of Relativity with Quantum Mechanics: Quantum Field Theory...
- The SM provides a unifying framework for 3 of the fundamental forces (strong, weak, electromagnetic)...
- It has been spectacularly confirmed... the latest discovery being the BEH boson from LHC in 2012...
- More than 15 Nobel prizes since 1950...




- The model has 19 free parameters (10 masses, 3 gauge couplings, 5 angles, Higgs vev)... Particles: quarks & antiquarks, leptons & antileptons, gauge & Higgs bosons...
- What determines the free parameters? Why this particular pattern of gauge fields and multiplets?

# General theory of relativity

- The other great pillar of theoretical physics is the General theory of Relativity...

# General theory of relativity

- The other great pillar of theoretical physics is the General theory of Relativity...
- Despite its name, the theory is virtually a theory for the gravitational field...

# General theory of relativity

- The other great pillar of theoretical physics is the General theory of Relativity...
- Despite its name, the theory is virtually a theory for the gravitational field...
- Again, a series of impressive confirmations...

# General theory of relativity

- The other great pillar of theoretical physics is the General theory of Relativity...
- Despite its name, the theory is virtually a theory for the gravitational field...
- Again, a series of impressive confirmations... the latest being the discovery of gravitational waves from LIGO in 2016...

# General theory of relativity

- The other great pillar of theoretical physics is the General theory of Relativity...
- Despite its name, the theory is virtually a theory for the gravitational field...
- Again, a series of impressive confirmations... the latest being the discovery of gravitational waves from LIGO in 2016...
- Nobel prizes... (even more to come!)

1993 2011 2017

# General theory of relativity

- The other great pillar of theoretical physics is the General theory of Relativity...
- Despite its name, the theory is virtually a theory for the gravitational field...
- Again, a series of impressive confirmations... the latest being the discovery of gravitational waves from LIGO in 2016...
- Nobel prizes... (even more to come!)

1993 2011 2017

- Classical theory breaks down at the (unresolvable) singularities...

# General theory of relativity

- The other great pillar of theoretical physics is the General theory of Relativity...
- Despite its name, the theory is virtually a theory for the gravitational field...
- Again, a series of impressive confirmations... the latest being the discovery of gravitational waves from LIGO in 2016...
- Nobel prizes... (even more to come!)

1993 2011 2017

- Classical theory breaks down at the (unresolvable) singularities...
- Trying to combine gravity with quantum mechanics (Quantum Gravity) yields a non-renormalizable QFT...

# General theory of relativity

- The other great pillar of theoretical physics is the General theory of Relativity...
- Despite its name, the theory is virtually a theory for the gravitational field...
- Again, a series of impressive confirmations... the latest being the discovery of gravitational waves from LIGO in 2016...
- Nobel prizes... (even more to come!)

1993 2011 2017

- Classical theory breaks down at the (unresolvable) singularities...
- Trying to combine gravity with quantum mechanics (Quantum Gravity) yields a non-renormalizable QFT...
- Naturalness & hierarchy problems: e.g. why is the weak force so much stronger than gravity?

# String theory

- There have been many interesting ideas dealing with all these issues...

# String theory

- There have been many interesting ideas dealing with all these issues... For example: • GUTs...

# String theory

- There have been many interesting ideas dealing with all these issues... For example: • GUTs... • higher-dimensional spacetimes (Kaluza-Klein mechanism)...

# String theory

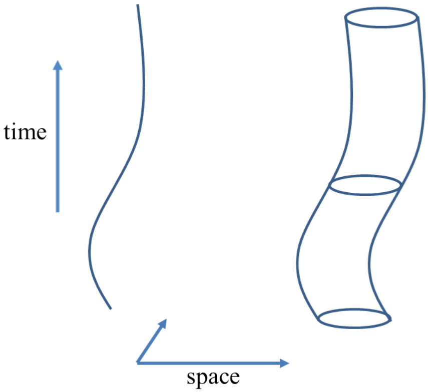
- There have been many interesting ideas dealing with all these issues... For example: • GUTs... • higher-dimensional spacetimes (Kaluza-Klein mechanism)... • supersymmetry...

# String theory

- There have been many interesting ideas dealing with all these issues... For example: • GUTs... • higher-dimensional spacetimes (Kaluza-Klein mechanism)... • supersymmetry...
- String theory combines the above ideas in a proposal that resolves the divergence problem...

# String theory

- There have been many interesting ideas dealing with all these issues... For example: • GUTs... • higher-dimensional spacetimes (Kaluza-Klein mechanism)... • supersymmetry...
- String theory combines the above ideas in a proposal that resolves the divergence problem... The idea is simple: spread out point particles to form strings!



# String theory

Key features of string theory:

- String theory contains a massless spin-2 state (graviton) whose interactions are those predicted by GR.

# String theory

Key features of string theory:

- String theory contains a massless spin-2 state (graviton) whose interactions are those predicted by GR.
- A quantum gravity theory without divergences!

# String theory

Key features of string theory:

- String theory contains a massless spin-2 state (graviton) whose interactions are those predicted by GR.
- A quantum gravity theory without divergences!
- Grand unification: string theory groups are large enough to contain the SM.

# String theory

Key features of string theory:

- String theory contains a massless spin-2 state (graviton) whose interactions are those predicted by GR.
- A quantum gravity theory without divergences!
- Grand unification: string theory groups are large enough to contain the SM.
- Extra dimensions: usually 4 large and 6 small compact dimensions.

# String theory

Key features of string theory:

- String theory contains a massless spin-2 state (graviton) whose interactions are those predicted by GR.
- A quantum gravity theory without divergences!
- Grand unification: string theory groups are large enough to contain the SM.
- Extra dimensions: usually 4 large and 6 small compact dimensions.
- Supersymmetry: either manifest or spontaneously broken...

# String theory

Key features of string theory:

- String theory contains a massless spin-2 state (graviton) whose interactions are those predicted by GR.
- A quantum gravity theory without divergences!
- Grand unification: string theory groups are large enough to contain the SM.
- Extra dimensions: usually 4 large and 6 small compact dimensions.
- Supersymmetry: either manifest or spontaneously broken...
- Chiral gauge couplings allowed in string theory.

# String theory

Key features of string theory:

- String theory contains a massless spin-2 state (graviton) whose interactions are those predicted by GR.
- A quantum gravity theory without divergences!
- Grand unification: string theory groups are large enough to contain the SM.
- Extra dimensions: usually 4 large and 6 small compact dimensions.
- Supersymmetry: either manifest or spontaneously broken...
- Chiral gauge couplings allowed in string theory.
- No free parameters!

# String theory

Key features of string theory:

- String theory contains a massless spin-2 state (graviton) whose interactions are those predicted by GR.
- A quantum gravity theory without divergences!
- Grand unification: string theory groups are large enough to contain the SM.
- Extra dimensions: usually 4 large and 6 small compact dimensions.
- Supersymmetry: either manifest or spontaneously broken...
- Chiral gauge couplings allowed in string theory.
- No free parameters!
- Uniqueness: no freedom in choosing gauge group & representations

# String theory

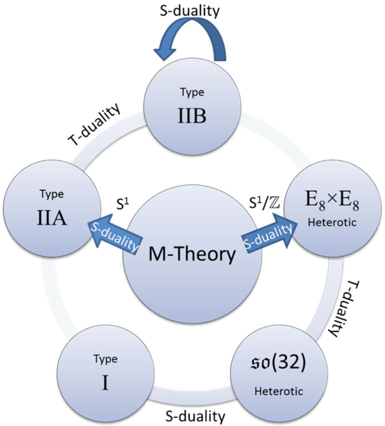
Key features of string theory:

- String theory contains a massless spin-2 state (graviton) whose interactions are those predicted by GR.
- A quantum gravity theory without divergences!
- Grand unification: string theory groups are large enough to contain the SM.
- Extra dimensions: usually 4 large and 6 small compact dimensions.
- Supersymmetry: either manifest or spontaneously broken...
- Chiral gauge couplings allowed in string theory.
- No free parameters!
- Uniqueness: no freedom in choosing gauge group & representations
- However: no experimental verification yet... string phenomenology in its infancy...

# M-theory

- By the end of the first superstring revolution (1984-1994), five different 10-dimensional superstring theories had emerged:

  Types I, II (IIA, IIB), Heterotic ($\mathfrak{so}(32)$, $E_8 \times E_8$).

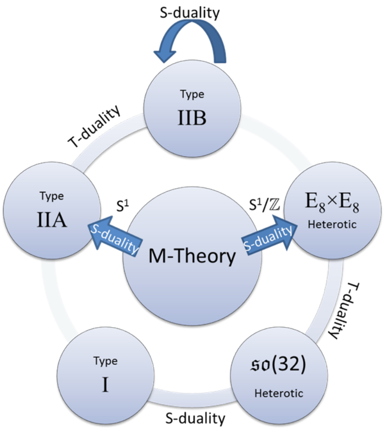


# M-theory

- By the end of the first superstring revolution (1984-1994), five different 10-dimensional superstring theories had emerged:

  Types I, II (IIA, IIB), Heterotic ($\mathfrak{so}(32)$, $E_8 \times E_8$).

- During the second superstring revolution (1994-2003), it was found that the 5 superstring theories are linked with dualities (T-duality, S-duality, U-duality, mirror symmetry)...



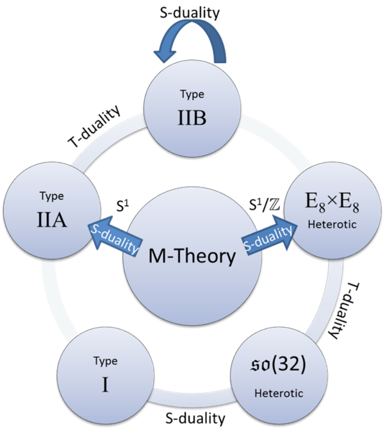
# M-theory

- By the end of the first superstring revolution (1984-1994), five different 10-dimensional superstring theories had emerged:

  Types I, II (IIA, IIB), Heterotic ($\mathfrak{so}(32)$, $E_8 \times E_8$).

- During the second superstring revolution (1994-2003), it was found that the 5 superstring theories are linked with dualities (T-duality, S-duality, U-duality, mirror symmetry)...

- What is more, it was realized that all five 10-dimensional superstring theories were just limiting cases of an 11-dimensional theory...



# M-theory

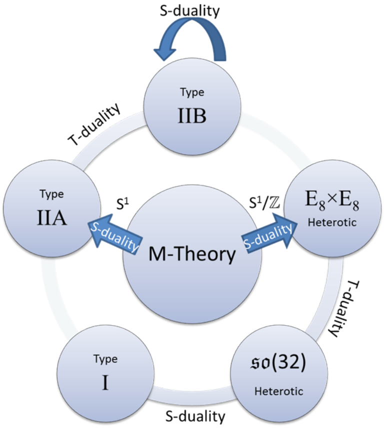
- By the end of the first superstring revolution (1984-1994), five different 10-dimensional superstring theories had emerged:

  Types I, II (IIA, IIB), Heterotic ($\mathfrak{so}(32)$, $E_8 \times E_8$).

- During the second superstring revolution (1994-2003), it was found that the 5 superstring theories are linked with dualities (T-duality, S-duality, U-duality, mirror symmetry)...

- What is more, it was realized that all five 10-dimensional superstring theories were just limiting cases of an 11-dimensional theory...

- This theory has been called "M-theory" and it is obtained in the strong-coupling limit ($g_s \to \infty$) of (IIA) superstring theory...

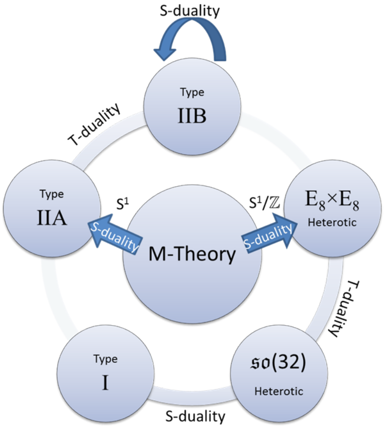


# M-theory

- By the end of the first superstring revolution (1984-1994), five different 10-dimensional superstring theories had emerged:

  Types I, II (IIA, IIB), Heterotic ($\mathfrak{so}(32)$, $E_8 \times E_8$).

- During the second superstring revolution (1994-2003), it was found that the 5 superstring theories are linked with dualities (T-duality, S-duality, U-duality, mirror symmetry)...
- What is more, it was realized that all five 10-dimensional superstring theories were just limiting cases of an 11-dimensional theory...
- This theory has been called "M-theory" and it is obtained in the strong-coupling limit ($g_s \to \infty$) of (IIA) superstring theory...
- The letter "M" stands for "magic, mystery and matrix" according to one of its founders, Edward Witten...

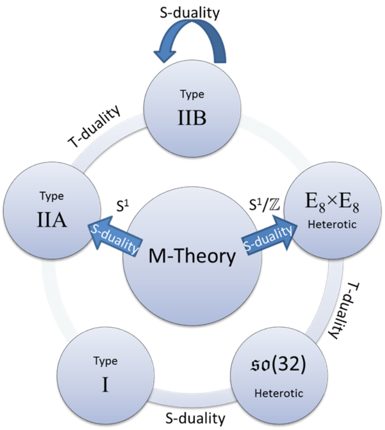


# M-theory

- By the end of the first superstring revolution (1984-1994), five different 10-dimensional superstring theories had emerged:

  Types I, II (IIA, IIB), Heterotic ($\mathfrak{so}(32)$, $E_8 \times E_8$).

- During the second superstring revolution (1994-2003), it was found that the 5 superstring theories are linked with dualities (T-duality, S-duality, U-duality, mirror symmetry)...
- What is more, it was realized that all five 10-dimensional superstring theories were just limiting cases of an 11-dimensional theory...
- This theory has been called "M-theory" and it is obtained in the strong-coupling limit ($g_s \to \infty$) of (IIA) superstring theory...
- The letter "M" stands for "magic, mystery and matrix" according to one of its founders, Edward Witten...
- Others have associated "M" with "membranes"...

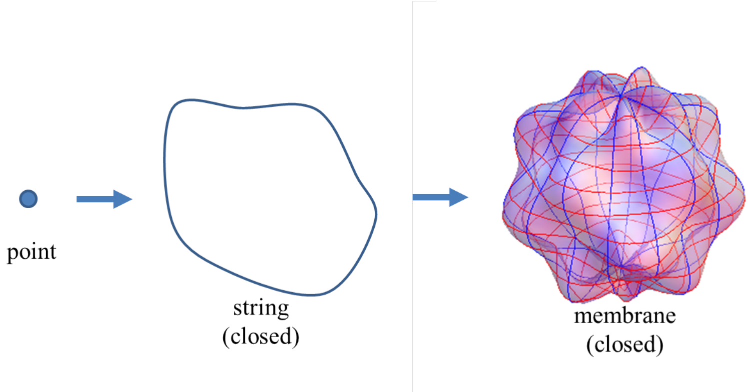


# Membranes

- The idea behind membrane theory is simple: replace 1-dimensional strings with 2-dimensional membranes...

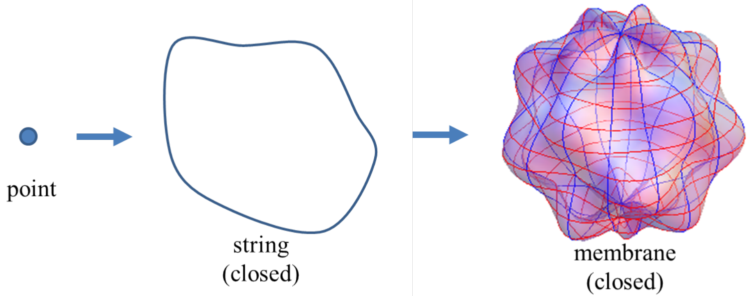
# Membranes

- The idea behind membrane theory is simple: replace 1-dimensional strings with 2-dimensional membranes...



# Membranes

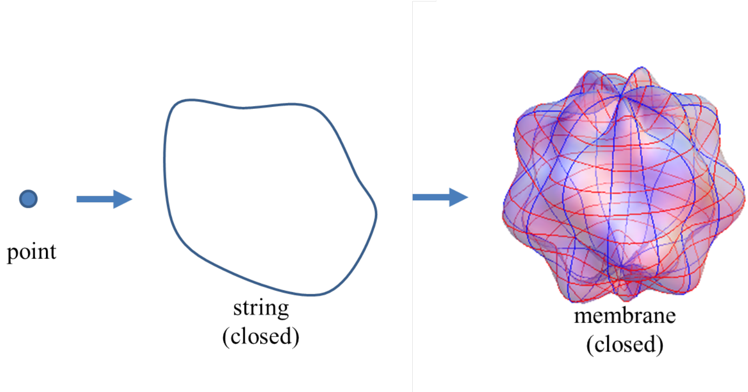
- The idea behind membrane theory is simple: replace 1-dimensional strings with 2-dimensional membranes...




- Like strings, membranes can be supersymmetrized... we thus obtain supermembranes...

# Membranes

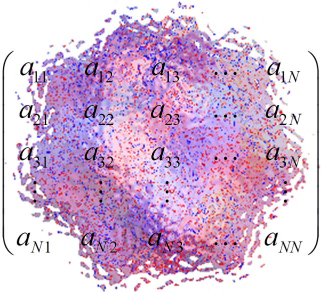
- The idea behind membrane theory is simple: replace 1-dimensional strings with 2-dimensional membranes...




- Like strings, membranes can be supersymmetrized... we thus obtain supermembranes...
- There are reasons to believe that membranes (or "M2-branes") are the fundamental objects of 11-dimensional M-theory, just like strings are the fundamental objects of 10-dimensional string theory...

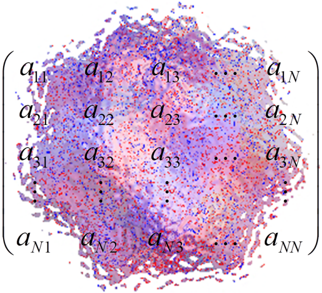
# Matrix theory

- According to the matrix theory conjecture of Banks, Fischler, Shenker and Susskind (BFSS, 1996), a theory of matrix-quantized supermembranes provides a realization of M-theory in flat spacetime...



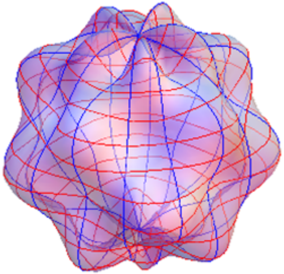
# Matrix theory

- According to the matrix theory conjecture of Banks, Fischler, Shenker and Susskind (BFSS, 1996), a theory of matrix-quantized supermembranes provides a realization of M-theory in flat spacetime...
- In the matrix-model language, membranes are fuzzy objects represented by $N \times N$ matrices...



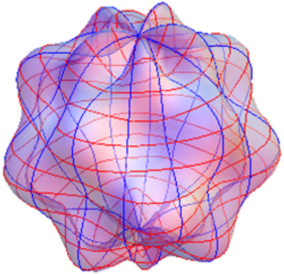
# Matrix theory

- According to the matrix theory conjecture of Banks, Fischler, Shenker and Susskind (BFSS, 1996), a theory of matrix-quantized supermembranes provides a realization of M-theory in flat spacetime...
- In the matrix-model language, membranes are fuzzy objects represented by $N \times N$ matrices...
- In the limit of large matrix dimensions ($N \to \infty$), matrix models are known to reduce to membrane theories...



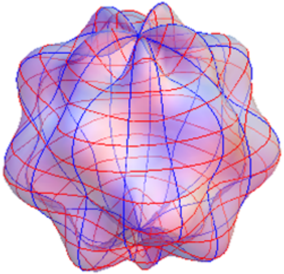
# Matrix theory

- According to the matrix theory conjecture of Banks, Fischler, Shenker and Susskind (BFSS, 1996), a theory of matrix-quantized supermembranes provides a realization of M-theory in flat space-time...
- In the matrix-model language, membranes are fuzzy objects represented by $N \times N$ matrices...
- In the limit of large matrix dimensions ($N \to \infty$), matrix models are known to reduce to membrane theories...
- In 2002, Berenstein, Maldacena and Nastase (BMN) proposed a reformulation of the BFSS matrix model on a particular type of weakly curved space-times that are known as plane-waves...



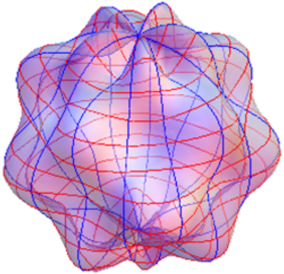
# Matrix theory

- According to the matrix theory conjecture of Banks, Fischler, Shenker and Susskind (BFSS, 1996), a theory of matrix-quantized supermembranes provides a realization of M-theory in flat space-time...
- In the matrix-model language, membranes are fuzzy objects represented by $N \times N$ matrices...
- In the limit of large matrix dimensions ($N \to \infty$), matrix models are known to reduce to membrane theories...
- In 2002, Berenstein, Maldacena and Nastase (BMN) proposed a reformulation of the BFSS matrix model on a particular type of weakly curved space-times that are known as plane-waves...
- The BMN matrix model is a deformation of the BFSS matrix model by mass terms and a flux (Myers) term...



# Matrix theory

- According to the matrix theory conjecture of Banks, Fischler, Shenker and Susskind (BFSS, 1996), a theory of matrix-quantized supermembranes provides a realization of M-theory in flat space-time...
- In the matrix-model language, membranes are fuzzy objects represented by $N \times N$ matrices...
- In the limit of large matrix dimensions ($N \to \infty$), matrix models are known to reduce to membrane theories...
- In 2002, Berenstein, Maldacena and Nastase (BMN) proposed a reformulation of the BFSS matrix model on a particular type of weakly curved space-times that are known as plane-waves...
- The BMN matrix model is a deformation of the BFSS matrix model by mass terms and a flux (Myers) term...
- In the large-$N$ limit, the BMN matrix model is again known to reduce to a theory of membranes inside an 11-dimensional plane-wave background...



# Membranes in plane-wave backgrounds

# General setup

- Consider the 11-dimensional maximally supersymmetric plane-wave background:

$$ds^2 = -2dx^+dx^- - \left[\frac{\mu^2}{9}\sum_{i=1}^{3} x^i x^i + \frac{\mu^2}{36}\sum_{j=1}^{6} y^j y^j\right] dx^+dx^+ + dx^i dx^i + dy^j dy^j, \qquad F_{123+} = \mu.$$

# General setup

- Consider the 11-dimensional maximally supersymmetric plane-wave background:

$$ds^2 = -2dx^+dx^- - \left[\frac{\mu^2}{9}\sum_{i=1}^{3} x^i x^i + \frac{\mu^2}{36}\sum_{j=1}^{6} y^j y^j\right] dx^+dx^+ + dx^i dx^i + dy^j dy^j, \qquad F_{123+} = \mu.$$

- The Hamiltonian of a bosonic relativistic membrane in the above background reads

$$\boxed{H = \frac{T}{2}\int d^2\sigma \left[p_x^2 + p_y^2 + \frac{1}{2}\left\{x^i, x^j\right\}^2 + \frac{1}{2}\left\{y^i, y^j\right\}^2 + \left\{x^i, y^j\right\}^2 + \frac{\mu^2 x^2}{9} + \frac{\mu^2 y^2}{36} - \frac{\mu}{3}\epsilon_{ijk}\left\{x^i, x^j\right\} x^k\right]},$$

in the so-called light-cone gauge ($x_+ = \tau$).

# General setup

- Consider the 11-dimensional maximally supersymmetric plane-wave background:

$$ds^2 = -2dx^+dx^- - \left[\frac{\mu^2}{9}\sum_{i=1}^{3} x^i x^i + \frac{\mu^2}{36}\sum_{j=1}^{6} y^j y^j\right] dx^+ dx^+ + dx^i dx^i + dy^j dy^j, \qquad F_{123+} = \mu.$$

- The Hamiltonian of a bosonic relativistic membrane in the above background reads

$$\boxed{H = \frac{T}{2}\int d^2\sigma \left[p_x^2 + p_y^2 + \frac{1}{2}\left\{x^i, x^j\right\}^2 + \frac{1}{2}\left\{y^i, y^j\right\}^2 + \left\{x^i, y^j\right\}^2 + \frac{\mu^2 x^2}{9} + \frac{\mu^2 y^2}{36} - \frac{\mu}{3}\epsilon_{ijk}\left\{x^i, x^j\right\} x^k\right]},$$

in the so-called light-cone gauge ($x_+ = \tau$).

- The corresponding equations of motion and the Gauß-law constraint are given by:

$$\ddot{x}_i = \left\{\left\{x_i, x_j\right\}, x_j\right\} + \left\{\left\{x_i, y_j\right\}, y_j\right\} - \frac{\mu^2}{9} x_i + \frac{\mu}{2}\epsilon_{ijk}\left\{x_j, x_k\right\}, \qquad \sum_{i=1}^{3}\left\{\dot{x}^i, x^i\right\} + \sum_{j=1}^{6}\left\{\dot{y}^j, y^j\right\} = 0$$

$$\ddot{y}_i = \left\{\left\{y_i, y_j\right\}, y_j\right\} + \left\{\left\{y_i, x_j\right\}, x_j\right\} - \frac{\mu^2}{36} y_i.$$

# The ansatz

- The following $\mathfrak{so}(3)$-invariant ansatz automatically satisfies the Gauß-law constraint:

$$x_i = \tilde{u}_i(\tau)\, e_i, \qquad y_j = \tilde{v}_j(\tau)\, e_j, \qquad y_{j+3} = \tilde{w}_j(\tau)\, e_j, \qquad i, j = 1, 2, 3.$$

Collins-Tucker (1976), Harmark-Savvidy

# The ansatz

- The following $\mathfrak{so}\,(3)$-invariant ansatz automatically satisfies the Gauß-law constraint:

$$x_i = \tilde{u}_i\,(\tau)\,e_i\,, \qquad y_j = \tilde{v}_j\,(\tau)\,e_j\,, \qquad y_{j+3} = \tilde{w}_j\,(\tau)\,e_j\,, \qquad i,j = 1,2,3.$$

Collins-Tucker (1976), Harmark-Savvidy

The coordinates $e_i$ are given by

$$(e_1\,, e_2\,, e_3) \equiv (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta), \qquad \phi \in [0, 2\pi), \quad \theta \in [0, \pi],$$

satisfy the $\mathfrak{so}\,(3)$ Poisson algebra and are orthonormal:

$$\{e_i, e_j\} = \epsilon_{ijk}\,e_k\,, \qquad \int e_i\,e_j\,d^2\sigma = \frac{4\pi}{3}\,\delta_{ij}.$$

# The ansatz

- The following $\mathfrak{so}(3)$-invariant ansatz automatically satisfies the Gauß-law constraint:

$$x_i = \tilde{u}_i(\tau)\, e_i, \qquad y_j = \tilde{v}_j(\tau)\, e_j, \qquad y_{j+3} = \tilde{w}_j(\tau)\, e_j, \qquad i, j = 1, 2, 3.$$

Collins-Tucker (1976), Harmark-Savvidy

The coordinates $e_i$ are given by

$$(e_1, e_2, e_3) \equiv (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta), \qquad \phi \in [0, 2\pi), \quad \theta \in [0, \pi],$$

satisfy the $\mathfrak{so}(3)$ Poisson algebra and are orthonormal:

$$\{e_i, e_j\} = \epsilon_{ijk}\, e_k, \qquad \int e_i\, e_j\, d^2\sigma = \frac{4\pi}{3}\, \delta_{ij}.$$

- As we will see, the above ansatz leads to an interesting dynamical system with stable and unstable solutions that describe rotating and pulsating membranes of spherical topology.

# The ansatz

- The following $\mathfrak{so}(3)$-invariant ansatz automatically satisfies the Gauß-law constraint:

$$x_i = \tilde{u}_i(\tau)\, e_i, \qquad y_j = \tilde{v}_j(\tau)\, e_j, \qquad y_{j+3} = \tilde{w}_j(\tau)\, e_j, \qquad i, j = 1, 2, 3.$$

Collins-Tucker (1976), Harmark-Savvidy

The coordinates $e_i$ are given by

$$(e_1, e_2, e_3) \equiv (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta), \qquad \phi \in [0, 2\pi), \quad \theta \in [0, \pi],$$

satisfy the $\mathfrak{so}(3)$ Poisson algebra and are orthonormal:

$$\{e_i, e_j\} = \epsilon_{ijk}\, e_k, \qquad \int e_i\, e_j\, d^2\sigma = \frac{4\pi}{3}\, \delta_{ij}.$$

- As we will see, the above ansatz leads to an interesting dynamical system with stable and unstable solutions that describe rotating and pulsating membranes of spherical topology.
- Similar work in flat space has previously been carried out by (Axenides-Floratos, 2007).

# The Hamiltonian system

- Here's the Hamiltonian of the membrane:

$$H = \frac{2\pi T}{3}\Big[\tilde{p}_u^2 + \tilde{p}_v^2 + \tilde{p}_w^2 + \tilde{u}_1^2\tilde{u}_2^2 + \tilde{u}_2^2\tilde{u}_3^2 + \tilde{u}_3^2\tilde{u}_1^2 + \tilde{r}_1^2\tilde{r}_2^2 + \tilde{r}_2^2\tilde{r}_3^2 + \tilde{r}_3^2\tilde{r}_1^2 + \tilde{u}_1^2\left(\tilde{r}_2^2 + \tilde{r}_3^2\right) + \tilde{u}_2^2\left(\tilde{r}_3^2 + \tilde{r}_1^2\right) +$$

$$+\tilde{u}_3^2\left(\tilde{r}_1^2 + \tilde{r}_2^2\right) + \frac{\mu^2}{9}\left(\tilde{u}_1^2 + \tilde{u}_2^2 + \tilde{u}_3^2\right) + \frac{\mu^2}{36}\left(\tilde{r}_1^2 + \tilde{r}_2^2 + \tilde{r}_3^2\right) - 2\mu\tilde{u}_1\tilde{u}_2\tilde{u}_3\Big], \quad \tilde{r}_j^2 \equiv \tilde{v}_j^2 + \tilde{w}_j^2, \ j = 1, 2, 3.$$

# The Hamiltonian system

- Here's the Hamiltonian of the membrane:

$$H = \frac{2\pi T}{3}\Big[\tilde{p}_u^2 + \tilde{p}_v^2 + \tilde{p}_w^2 + \tilde{u}_1^2\tilde{u}_2^2 + \tilde{u}_2^2\tilde{u}_3^2 + \tilde{u}_3^2\tilde{u}_1^2 + \tilde{r}_1^2\tilde{r}_2^2 + \tilde{r}_2^2\tilde{r}_3^2 + \tilde{r}_3^2\tilde{r}_1^2 + \tilde{u}_1^2\left(\tilde{r}_2^2 + \tilde{r}_3^2\right) + \tilde{u}_2^2\left(\tilde{r}_3^2 + \tilde{r}_1^2\right) +$$

$$+\tilde{u}_3^2\left(\tilde{r}_1^2 + \tilde{r}_2^2\right) + \frac{\mu^2}{9}\left(\tilde{u}_1^2 + \tilde{u}_2^2 + \tilde{u}_3^2\right) + \frac{\mu^2}{36}\left(\tilde{r}_1^2 + \tilde{r}_2^2 + \tilde{r}_3^2\right) - 2\mu\tilde{u}_1\tilde{u}_2\tilde{u}_3\Big], \quad \tilde{r}_j^2 \equiv \tilde{v}_j^2 + \tilde{w}_j^2, \ j = 1, 2, 3.$$

- The Hamiltonian has an obvious $SO(2) \times SO(2) \times SO(2)$ symmetry in the coordinates $\tilde{v}_i$ and $\tilde{w}_i$ so that any solution will preserve three $SO(2)$ angular momenta $\ell_i$ $(i = 1, 2, 3)$. The kinetic terms are written as:

$$\tilde{p}_v^2 + \tilde{p}_w^2 = \sum_{i=1}^{3}\left(\dot{\tilde{r}}_i^2 + \frac{\ell_i^2}{\tilde{r}_i^2}\right).$$

# The Hamiltonian system

- Here's the Hamiltonian of the membrane:

$$H = \frac{2\pi T}{3}\Big[\tilde{p}_u^2 + \tilde{p}_v^2 + \tilde{p}_w^2 + \tilde{u}_1^2\tilde{u}_2^2 + \tilde{u}_2^2\tilde{u}_3^2 + \tilde{u}_3^2\tilde{u}_1^2 + \tilde{r}_1^2\tilde{r}_2^2 + \tilde{r}_2^2\tilde{r}_3^2 + \tilde{r}_3^2\tilde{r}_1^2 + \tilde{u}_1^2\left(\tilde{r}_2^2 + \tilde{r}_3^2\right) + \tilde{u}_2^2\left(\tilde{r}_3^2 + \tilde{r}_1^2\right) + $$
$$+ \tilde{u}_3^2\left(\tilde{r}_1^2 + \tilde{r}_2^2\right) + \frac{\mu^2}{9}\left(\tilde{u}_1^2 + \tilde{u}_2^2 + \tilde{u}_3^2\right) + \frac{\mu^2}{36}\left(\tilde{r}_1^2 + \tilde{r}_2^2 + \tilde{r}_3^2\right) - 2\mu\tilde{u}_1\tilde{u}_2\tilde{u}_3\Big], \quad \tilde{r}_j^2 \equiv \tilde{v}_j^2 + \tilde{w}_j^2, \ j = 1, 2, 3.$$

- The Hamiltonian has an obvious $SO(2)\times SO(2)\times SO(2)$ symmetry in the coordinates $\tilde{v}_i$ and $\tilde{w}_i$ so that any solution will preserve three $SO(2)$ angular momenta $\ell_i$ $(i = 1, 2, 3)$. The kinetic terms are written as:

$$\tilde{p}_v^2 + \tilde{p}_w^2 = \sum_{i=1}^{3}\left(\dot{\tilde{r}}_i^2 + \frac{\ell_i^2}{\tilde{r}_i^2}\right).$$

- The effective potential of the membrane becomes:

$$V_{\text{eff}} = \frac{2\pi T}{3}\Big[\frac{\ell_1^2}{\tilde{r}_1^2} + \frac{\ell_2^2}{\tilde{r}_2^2} + \frac{\ell_3^2}{\tilde{r}_3^2} + \tilde{u}_1^2\tilde{u}_2^2 + \tilde{u}_2^2\tilde{u}_3^2 + \tilde{u}_3^2\tilde{u}_1^2 + \tilde{r}_1^2\tilde{r}_2^2 + \tilde{r}_2^2\tilde{r}_3^2 + \tilde{r}_3^2\tilde{r}_1^2 + \tilde{u}_1^2\left(\tilde{r}_2^2 + \tilde{r}_3^2\right) + \tilde{u}_2^2\left(\tilde{r}_3^2 + \tilde{r}_1^2\right) + $$
$$+ \tilde{u}_3^2\left(\tilde{r}_1^2 + \tilde{r}_2^2\right) + \frac{\mu^2}{9}\left(\tilde{u}_1^2 + \tilde{u}_2^2 + \tilde{u}_3^2\right) + \frac{\mu^2}{36}\left(\tilde{r}_1^2 + \tilde{r}_2^2 + \tilde{r}_3^2\right) - 2\mu\tilde{u}_1\tilde{u}_2\tilde{u}_3\Big].$$

# The Hamiltonian system

- Here's the Hamiltonian of the membrane:

$$H = \frac{2\pi T}{3}\Big[\tilde{p}_u^2 + \tilde{p}_v^2 + \tilde{p}_w^2 + \tilde{u}_1^2\tilde{u}_2^2 + \tilde{u}_2^2\tilde{u}_3^2 + \tilde{u}_3^2\tilde{u}_1^2 + \tilde{r}_1^2\tilde{r}_2^2 + \tilde{r}_2^2\tilde{r}_3^2 + \tilde{r}_3^2\tilde{r}_1^2 + \tilde{u}_1^2\left(\tilde{r}_2^2 + \tilde{r}_3^2\right) + \tilde{u}_2^2\left(\tilde{r}_3^2 + \tilde{r}_1^2\right) + \\ + \tilde{u}_3^2\left(\tilde{r}_1^2 + \tilde{r}_2^2\right) + \frac{\mu^2}{9}\left(\tilde{u}_1^2 + \tilde{u}_2^2 + \tilde{u}_3^2\right) + \frac{\mu^2}{36}\left(\tilde{r}_1^2 + \tilde{r}_2^2 + \tilde{r}_3^2\right) - 2\mu\tilde{u}_1\tilde{u}_2\tilde{u}_3\Big], \quad \tilde{r}_j^2 \equiv \tilde{v}_j^2 + \tilde{w}_j^2, \ j = 1, 2, 3.$$

- The Hamiltonian has an obvious $SO(2)\times SO(2)\times SO(2)$ symmetry in the coordinates $\tilde{v}_i$ and $\tilde{w}_i$ so that any solution will preserve three $SO(2)$ angular momenta $\ell_i$ $(i = 1, 2, 3)$. The kinetic terms are written as:

$$\tilde{p}_v^2 + \tilde{p}_w^2 = \sum_{i=1}^{3}\left(\dot{\tilde{r}}_i^2 + \frac{\ell_i^2}{\tilde{r}_i^2}\right).$$

- The effective potential of the membrane becomes:

$$V_{\text{eff}} = \frac{2\pi T}{3}\Big[\frac{\ell_1^2}{\tilde{r}_1^2} + \frac{\ell_2^2}{\tilde{r}_2^2} + \frac{\ell_3^2}{\tilde{r}_3^2} + \tilde{u}_1^2\tilde{u}_2^2 + \tilde{u}_2^2\tilde{u}_3^2 + \tilde{u}_3^2\tilde{u}_1^2 + \tilde{r}_1^2\tilde{r}_2^2 + \tilde{r}_2^2\tilde{r}_3^2 + \tilde{r}_3^2\tilde{r}_1^2 + \tilde{u}_1^2\left(\tilde{r}_2^2 + \tilde{r}_3^2\right) + \tilde{u}_2^2\left(\tilde{r}_3^2 + \tilde{r}_1^2\right) + \\ + \tilde{u}_3^2\left(\tilde{r}_1^2 + \tilde{r}_2^2\right) + \frac{\mu^2}{9}\left(\tilde{u}_1^2 + \tilde{u}_2^2 + \tilde{u}_3^2\right) + \frac{\mu^2}{36}\left(\tilde{r}_1^2 + \tilde{r}_2^2 + \tilde{r}_3^2\right) - 2\mu\tilde{u}_1\tilde{u}_2\tilde{u}_3\Big].$$

- There are four distinct types of terms: • (1) $\ell$-terms (repulsive)

# The Hamiltonian system

- Here's the Hamiltonian of the membrane:

$$H = \frac{2\pi T}{3}\Big[\tilde{p}_u^2 + \tilde{p}_v^2 + \tilde{p}_w^2 + \tilde{u}_1^2\tilde{u}_2^2 + \tilde{u}_2^2\tilde{u}_3^2 + \tilde{u}_3^2\tilde{u}_1^2 + \tilde{r}_1^2\tilde{r}_2^2 + \tilde{r}_2^2\tilde{r}_3^2 + \tilde{r}_3^2\tilde{r}_1^2 + \tilde{u}_1^2\left(\tilde{r}_2^2+\tilde{r}_3^2\right) + \tilde{u}_2^2\left(\tilde{r}_3^2+\tilde{r}_1^2\right) + \\ + \tilde{u}_3^2\left(\tilde{r}_1^2+\tilde{r}_2^2\right) + \frac{\mu^2}{9}\left(\tilde{u}_1^2+\tilde{u}_2^2+\tilde{u}_3^2\right) + \frac{\mu^2}{36}\left(\tilde{r}_1^2+\tilde{r}_2^2+\tilde{r}_3^2\right) - 2\mu\tilde{u}_1\tilde{u}_2\tilde{u}_3\Big], \quad \tilde{r}_j^2 \equiv \tilde{v}_j^2 + \tilde{w}_j^2,\ j = 1,2,3.$$

- The Hamiltonian has an obvious $SO(2)\times SO(2)\times SO(2)$ symmetry in the coordinates $\tilde{v}_i$ and $\tilde{w}_i$ so that any solution will preserve three $SO(2)$ angular momenta $\ell_i$ $(i = 1,2,3)$. The kinetic terms are written as:

$$\tilde{p}_v^2 + \tilde{p}_w^2 = \sum_{i=1}^{3}\left(\dot{\tilde{r}}_i^2 + \frac{\ell_i^2}{\tilde{r}_i^2}\right).$$

- The effective potential of the membrane becomes:

$$V_{\text{eff}} = \frac{2\pi T}{3}\Big[\frac{\ell_1^2}{\tilde{r}_1^2} + \frac{\ell_2^2}{\tilde{r}_2^2} + \frac{\ell_3^2}{\tilde{r}_3^2} + \tilde{u}_1^2\tilde{u}_2^2 + \tilde{u}_2^2\tilde{u}_3^2 + \tilde{u}_3^2\tilde{u}_1^2 + \tilde{r}_1^2\tilde{r}_2^2 + \tilde{r}_2^2\tilde{r}_3^2 + \tilde{r}_3^2\tilde{r}_1^2 + \tilde{u}_1^2\left(\tilde{r}_2^2+\tilde{r}_3^2\right) + \tilde{u}_2^2\left(\tilde{r}_3^2+\tilde{r}_1^2\right) + \\ + \tilde{u}_3^2\left(\tilde{r}_1^2+\tilde{r}_2^2\right) + \frac{\mu^2}{9}\left(\tilde{u}_1^2+\tilde{u}_2^2+\tilde{u}_3^2\right) + \frac{\mu^2}{36}\left(\tilde{r}_1^2+\tilde{r}_2^2+\tilde{r}_3^2\right) - 2\mu\tilde{u}_1\tilde{u}_2\tilde{u}_3\Big].$$

- There are four distinct types of terms: • (1) $\ell$-terms (repulsive) • (2) quartic interaction terms (attractive)

# The Hamiltonian system

- Here's the Hamiltonian of the membrane:

$$H = \frac{2\pi T}{3}\Big[\tilde{p}_u^2 + \tilde{p}_v^2 + \tilde{p}_w^2 + \tilde{u}_1^2\tilde{u}_2^2 + \tilde{u}_2^2\tilde{u}_3^2 + \tilde{u}_3^2\tilde{u}_1^2 + \tilde{r}_1^2\tilde{r}_2^2 + \tilde{r}_2^2\tilde{r}_3^2 + \tilde{r}_3^2\tilde{r}_1^2 + \tilde{u}_1^2\left(\tilde{r}_2^2 + \tilde{r}_3^2\right) + \tilde{u}_2^2\left(\tilde{r}_3^2 + \tilde{r}_1^2\right) + \\ + \tilde{u}_3^2\left(\tilde{r}_1^2 + \tilde{r}_2^2\right) + \frac{\mu^2}{9}\left(\tilde{u}_1^2 + \tilde{u}_2^2 + \tilde{u}_3^2\right) + \frac{\mu^2}{36}\left(\tilde{r}_1^2 + \tilde{r}_2^2 + \tilde{r}_3^2\right) - 2\mu\tilde{u}_1\tilde{u}_2\tilde{u}_3\Big], \quad \tilde{r}_j^2 \equiv \tilde{v}_j^2 + \tilde{w}_j^2, \ j = 1, 2, 3.$$

- The Hamiltonian has an obvious $SO(2) \times SO(2) \times SO(2)$ symmetry in the coordinates $\tilde{v}_i$ and $\tilde{w}_i$ so that any solution will preserve three $SO(2)$ angular momenta $\ell_i$ $(i = 1, 2, 3)$. The kinetic terms are written as:

$$\tilde{p}_v^2 + \tilde{p}_w^2 = \sum_{i=1}^{3}\left(\dot{\tilde{r}}_i^2 + \frac{\ell_i^2}{\tilde{r}_i^2}\right).$$

- The effective potential of the membrane becomes:

$$V_{\text{eff}} = \frac{2\pi T}{3}\Big[\frac{\ell_1^2}{\tilde{r}_1^2} + \frac{\ell_2^2}{\tilde{r}_2^2} + \frac{\ell_3^2}{\tilde{r}_3^2} + \tilde{u}_1^2\tilde{u}_2^2 + \tilde{u}_2^2\tilde{u}_3^2 + \tilde{u}_3^2\tilde{u}_1^2 + \tilde{r}_1^2\tilde{r}_2^2 + \tilde{r}_2^2\tilde{r}_3^2 + \tilde{r}_3^2\tilde{r}_1^2 + \tilde{u}_1^2\left(\tilde{r}_2^2 + \tilde{r}_3^2\right) + \tilde{u}_2^2\left(\tilde{r}_3^2 + \tilde{r}_1^2\right) + \\ + \tilde{u}_3^2\left(\tilde{r}_1^2 + \tilde{r}_2^2\right) + \frac{\mu^2}{9}\left(\tilde{u}_1^2 + \tilde{u}_2^2 + \tilde{u}_3^2\right) + \frac{\mu^2}{36}\left(\tilde{r}_1^2 + \tilde{r}_2^2 + \tilde{r}_3^2\right) - 2\mu\tilde{u}_1\tilde{u}_2\tilde{u}_3\Big].$$

- There are four distinct types of terms: • (1) $\ell$-terms (repulsive) • (2) quartic interaction terms (attractive) • (3) $\mu$-terms (attractive)

# The Hamiltonian system

- Here's the Hamiltonian of the membrane:

$$H = \frac{2\pi T}{3}\Big[\tilde{p}_u^2 + \tilde{p}_v^2 + \tilde{p}_w^2 + \tilde{u}_1^2\tilde{u}_2^2 + \tilde{u}_2^2\tilde{u}_3^2 + \tilde{u}_3^2\tilde{u}_1^2 + \tilde{r}_1^2\tilde{r}_2^2 + \tilde{r}_2^2\tilde{r}_3^2 + \tilde{r}_3^2\tilde{r}_1^2 + \tilde{u}_1^2\left(\tilde{r}_2^2 + \tilde{r}_3^2\right) + \tilde{u}_2^2\left(\tilde{r}_3^2 + \tilde{r}_1^2\right) + \\ + \tilde{u}_3^2\left(\tilde{r}_1^2 + \tilde{r}_2^2\right) + \frac{\mu^2}{9}\left(\tilde{u}_1^2 + \tilde{u}_2^2 + \tilde{u}_3^2\right) + \frac{\mu^2}{36}\left(\tilde{r}_1^2 + \tilde{r}_2^2 + \tilde{r}_3^2\right) - 2\mu\tilde{u}_1\tilde{u}_2\tilde{u}_3\Big], \quad \tilde{r}_j^2 \equiv \tilde{v}_j^2 + \tilde{w}_j^2, \; j = 1, 2, 3.$$

- The Hamiltonian has an obvious $SO(2)\times SO(2)\times SO(2)$ symmetry in the coordinates $\tilde{v}_i$ and $\tilde{w}_i$ so that any solution will preserve three $SO(2)$ angular momenta $\ell_i$ $(i = 1, 2, 3)$. The kinetic terms are written as:

$$\tilde{p}_v^2 + \tilde{p}_w^2 = \sum_{i=1}^{3}\left(\dot{\tilde{r}}_i^2 + \frac{\ell_i^2}{\tilde{r}_i^2}\right).$$

- The effective potential of the membrane becomes:

$$V_{\text{eff}} = \frac{2\pi T}{3}\left[\frac{\ell_1^2}{\tilde{r}_1^2} + \frac{\ell_2^2}{\tilde{r}_2^2} + \frac{\ell_3^2}{\tilde{r}_3^2} + \tilde{u}_1^2\tilde{u}_2^2 + \tilde{u}_2^2\tilde{u}_3^2 + \tilde{u}_3^2\tilde{u}_1^2 + \tilde{r}_1^2\tilde{r}_2^2 + \tilde{r}_2^2\tilde{r}_3^2 + \tilde{r}_3^2\tilde{r}_1^2 + \tilde{u}_1^2\left(\tilde{r}_2^2 + \tilde{r}_3^2\right) + \tilde{u}_2^2\left(\tilde{r}_3^2 + \tilde{r}_1^2\right) + \\ + \tilde{u}_3^2\left(\tilde{r}_1^2 + \tilde{r}_2^2\right) + \frac{\mu^2}{9}\left(\tilde{u}_1^2 + \tilde{u}_2^2 + \tilde{u}_3^2\right) + \frac{\mu^2}{36}\left(\tilde{r}_1^2 + \tilde{r}_2^2 + \tilde{r}_3^2\right) - 2\mu\tilde{u}_1\tilde{u}_2\tilde{u}_3\right].$$

- There are four distinct types of terms: • (1) $\ell$-terms (repulsive) • (2) quartic interaction terms (attractive) • (3) $\mu$-terms (attractive) • (4) cubic Myers/flux terms (repulsive).

# Membranes as dynamical systems

# Fully asymmetric membrane

# The $SO(3)$ sector

- We will only consider the simplest possible case where the $SO(6)$ variables $\tilde{v}_i$ and $\tilde{w}_i$ are set to zero:

$$\tilde{v}_i = \tilde{w}_i = 0, \qquad i = 1, 2, 3.$$

# The $SO(3)$ sector

- We will only consider the simplest possible case where the $SO(6)$ variables $\tilde{v}_i$ and $\tilde{w}_i$ are set to zero:

$$\tilde{v}_i = \tilde{w}_i = 0, \qquad i = 1, 2, 3.$$

We also scale-out the mass parameter $\mu$ by setting:

$$t = \mu\tau, \qquad \tilde{u}_i = \mu u_i.$$

# The $SO(3)$ sector

- We will only consider the simplest possible case where the $SO(6)$ variables $\tilde{v}_i$ and $\tilde{w}_i$ are set to zero:

$$\tilde{v}_i = \tilde{w}_i = 0, \qquad i = 1, 2, 3.$$

  We also scale-out the mass parameter $\mu$ by setting:

$$t = \mu\tau, \qquad \tilde{u}_i = \mu u_i.$$

- The effective potential of the membrane then becomes:

$$V_{\text{eff}} = \frac{2\pi T \mu^4}{3} \left[ u_1^2 u_2^2 + u_2^2 u_3^2 + u_1^2 u_3^2 + \frac{1}{9}\left(u_1^2 + u_2^2 + u_3^2\right) - 2 u_1 u_2 u_3 \right].$$

# The $SO(3)$ sector

- We will only consider the simplest possible case where the $SO(6)$ variables $\tilde{v}_i$ and $\tilde{w}_i$ are set to zero:

$$\tilde{v}_i = \tilde{w}_i = 0, \qquad i = 1, 2, 3.$$

We also scale-out the mass parameter $\mu$ by setting:

$$t = \mu\tau, \qquad \tilde{u}_i = \mu u_i.$$

- The effective potential of the membrane then becomes:

$$V_{\text{eff}} = \frac{2\pi T \mu^4}{3}\left[u_1^2 u_2^2 + u_2^2 u_3^2 + u_1^2 u_3^2 + \frac{1}{9}\left(u_1^2 + u_2^2 + u_3^2\right) - 2u_1 u_2 u_3\right].$$

- This is a particular instance of the generalized 3-dimensional Hénon-Heiles potential:

$$V_{\text{H-H}} = \frac{1}{2}\left(u_1^2 + u_2^2 + u_3^2\right) + K_3 \cdot u_1 u_2 u_3 + K_0\left(u_1^2 + u_2^2 + u_3^2\right)^2 + K_4\left(u_1^4 + u_2^4 + u_3^4\right),$$

Efstathiou-Sadovskií (2004)

with $K_3 = -9$, $K_0 = -K_4 = 9/4$.

# The $SO(3)$ sector

- Here are the critical points of the $SO(3)$ effective potential:

$$\mathbf{u}_0 = 0, \qquad \mathbf{u}_{1/6} = \frac{1}{6} \cdot (1,1,1)\,, \qquad \mathbf{u}_{1/3} = \frac{1}{3} \cdot (1,1,1)\,.$$

# The $SO(3)$ sector

- Here are the critical points of the $SO(3)$ effective potential:

$$\mathbf{u}_0 = 0, \qquad \mathbf{u}_{1/6} = \frac{1}{6} \cdot (1, 1, 1), \qquad \mathbf{u}_{1/3} = \frac{1}{3} \cdot (1, 1, 1).$$

- 6 more critical points can be obtained by flipping the sign of exactly two $u_i$'s. This is consistent with the manifest tetrahedral ($T_d$) symmetry of the effective potential.

# The $SO(3)$ sector

- Here are the critical points of the $SO(3)$ effective potential:

$$\mathbf{u}_0 = 0, \qquad \mathbf{u}_{1/6} = \frac{1}{6}\cdot(1,1,1)\,, \qquad \mathbf{u}_{1/3} = \frac{1}{3}\cdot(1,1,1)\,.$$

- 6 more critical points can be obtained by flipping the sign of exactly two $u_i$'s. This is consistent with the manifest tetrahedral ($T_d$) symmetry of the effective potential.
- $\mathbf{u}_0$ (a point-like membrane) and $\mathbf{u}_{1/3}$ (the Myers dielectric sphere) are global (degenerate) minima of the potential while $\mathbf{u}_{1/6}$ is a saddle point:

$$V_{\text{eff}}(0) = V_{\text{eff}}\left(\frac{1}{3}\right) = 0, \qquad V_{\text{eff}}\left(\frac{1}{6}\right) = \frac{2\pi T\mu^4}{6^4}.$$

# The $SO(3)$ sector

- Here are the critical points of the $SO(3)$ effective potential:

$$\mathbf{u}_0 = 0, \qquad \mathbf{u}_{1/6} = \frac{1}{6}\cdot(1,1,1)\,, \qquad \mathbf{u}_{1/3} = \frac{1}{3}\cdot(1,1,1)\,.$$

- 6 more critical points can be obtained by flipping the sign of exactly two $u_i$'s. This is consistent with the manifest tetrahedral ($T_d$) symmetry of the effective potential.
- $\mathbf{u}_0$ (a point-like membrane) and $\mathbf{u}_{1/3}$ (the Myers dielectric sphere) are global (degenerate) minima of the potential while $\mathbf{u}_{1/6}$ is a saddle point:

$$V_{\text{eff}}(0) = V_{\text{eff}}\left(\frac{1}{3}\right) = 0, \qquad V_{\text{eff}}\left(\frac{1}{6}\right) = \frac{2\pi T\mu^4}{6^4}.$$

- The Hamilton equations of motion become:

$$\begin{aligned}
\dot{u}_1 &= p_1, \qquad \dot{p}_1 = -\left[u_1\left(u_2^2+u_3^2\right) + \frac{u_1}{9} - u_2 u_3\right] \\
\dot{u}_2 &= p_2, \qquad \dot{p}_2 = -\left[u_2\left(u_3^2+u_1^2\right) + \frac{u_2}{9} - u_3 u_1\right] \\
\dot{u}_3 &= p_3, \qquad \dot{p}_3 = -\left[u_3\left(u_1^2+u_2^2\right) + \frac{u_3}{9} - u_1 u_2\right].
\end{aligned}$$

# Radial stability analysis

- By radially perturbing each of the 9 critical points as:

$$u_i = u_i^0 + \delta u_i\,(t)\,,$$

# Radial stability analysis

- By radially perturbing each of the 9 critical points as:

$$u_i = u_i^0 + \delta u_i(t),$$

we may confirm the conclusion derived above by examining the corresponding Hessian matrix, i.e. that $\mathbf{u}_0$ and $\mathbf{u}_{1/3}$ are global minima (positive-definite Hessian) and $\mathbf{u}_{1/6}$ is a saddle point (indefinite Hessian):

| critical point | eigenvalues $\lambda^2$ (#) | stability |
|---|---|---|
| $\mathbf{u}_0$ | $-\frac{1}{9}$ (3), $-\frac{1}{36}$ (6) | center (S) |
| $\mathbf{u}_{1/6}$ | $\frac{1}{18}$ (1), $-\frac{5}{18}$ (2), $-\frac{1}{12}$ (6) | saddle point |
| $\mathbf{u}_{1/3}$ | $-\frac{1}{9}$ (1), $-\frac{4}{9}$ (2), $-\frac{1}{4}$ (6) | center (S) |

Axenides-Floratos-G.L. (2017a)

# Angular stability analysis

- We may also perform more general (angular/multipole) perturbations of the following form:

$$x_i(t) = x_i^0 + \delta x_i(t), \qquad i = 1, 2, 3,$$

where $\delta x_i$ is expanded in spherical harmonics $Y_{jm}(\theta, \phi)$:

$$x_i(t) = \mu u_i(t)\, e_i, \qquad x_i^0 = \mu u_i^0 e_i, \qquad \delta x_i(t) = \mu \cdot \sum_{j=1}^{\infty} \sum_{m=-j}^{j} \eta_i^{jm}(t)\, Y_{jm}(\theta, \phi).$$

# Angular stability analysis

- We may also perform more general (angular/multipole) perturbations of the following form:

$$x_i(t) = x_i^0 + \delta x_i(t), \qquad i = 1, 2, 3,$$

where $\delta x_i$ is expanded in spherical harmonics $Y_{jm}(\theta, \phi)$:

$$x_i(t) = \mu u_i(t)\, e_i, \qquad x_i^0 = \mu u_i^0 e_i, \qquad \delta x_i(t) = \mu \cdot \sum_{j=1}^{\infty} \sum_{m=-j}^{j} \eta_i^{jm}(t)\, Y_{jm}(\theta, \phi).$$

- For each of the critical points $\mathbf{u}_0$, $\mathbf{u}_{1/6}$, $\mathbf{u}_{1/3}$ we find the eigenvalues (Axenides-Floratos-G.L., 2017b):

$$\mathbf{u}_0: \quad \lambda_P^2 = \lambda_\pm^2 = -\frac{1}{9}, \qquad \lambda_\theta^2 = -\frac{1}{36}$$

$$\mathbf{u}_{1/6}: \lambda_P^2 = 0, \qquad \lambda_+^2 = -\frac{1}{36}(j+1)(j+4), \qquad \lambda_-^2 = -\frac{j(j-3)}{36}, \qquad \lambda_\theta^2 = -\frac{1}{36}\left(j^2 + j + 1\right)$$

$$\mathbf{u}_{1/3}: \lambda_P^2 = 0, \qquad \lambda_+^2 = -\frac{1}{36}(j+1)^2, \qquad \lambda_-^2 = -\frac{j^2}{9}, \qquad \lambda_\theta^2 = -\frac{1}{36}(2j+1)^2,$$

with multiplicities $d_P = 2j+1$, $d_+ = 2j+3$, $d_- = 2j-1$ and $d_\theta = 6(2j+1)$, respectively.

# Angular stability analysis

- The critical point $\mathbf{u}_0$ (point-like membrane) is obviously stable.

# Angular stability analysis

- The critical point $\mathbf{u}_0$ (point-like membrane) is obviously stable.
- $\mathbf{u}_{1/3}$ has a zero mode of degeneracy $2d_P$ while all its other eigenvalues are stable for $j = 1, 2, \ldots$

# Angular stability analysis

- The critical point $\mathbf{u}_0$ (point-like membrane) is obviously stable.
- $\mathbf{u}_{1/3}$ has a zero mode of degeneracy $2d_P$ while all its other eigenvalues are stable for $j = 1, 2, \ldots$
- $\mathbf{u}_{1/6}$ has one $2d_P$-degenerate zero mode for every $j$ and a 10-fold degenerate zero mode for $j = 3$.

# Angular stability analysis

- The critical point $\mathbf{u}_0$ (point-like membrane) is obviously stable.
- $\mathbf{u}_{1/3}$ has a zero mode of degeneracy $2d_P$ while all its other eigenvalues are stable for $j = 1, 2, \ldots$
- $\mathbf{u}_{1/6}$ has one $2d_P$-degenerate zero mode for every $j$ and a 10-fold degenerate zero mode for $j = 3$. It is unstable for $j = 1$ (2-fold degenerate) and $j = 2$ (6-fold degenerate).

# Angular stability analysis

- The critical point $\mathbf{u}_0$ (point-like membrane) is obviously stable.
- $\mathbf{u}_{1/3}$ has a zero mode of degeneracy $2d_P$ while all its other eigenvalues are stable for $j = 1, 2, \ldots$
- $\mathbf{u}_{1/6}$ has one $2d_P$-degenerate zero mode for every $j$ and a 10-fold degenerate zero mode for $j = 3$. It is unstable for $j = 1$ (2-fold degenerate) and $j = 2$ (6-fold degenerate).
- The above results were first obtained by (Dasgupta, Sheikh-Jabbari, Raamsdonk, 2002) from the BMN matrix model point-of-view.

# Angular stability analysis

- The critical point $\mathbf{u}_0$ (point-like membrane) is obviously stable.
- $\mathbf{u}_{1/3}$ has a zero mode of degeneracy $2d_P$ while all its other eigenvalues are stable for $j = 1, 2, \ldots$
- $\mathbf{u}_{1/6}$ has one $2d_P$-degenerate zero mode for every $j$ and a 10-fold degenerate zero mode for $j = 3$. It is unstable for $j = 1$ (2-fold degenerate) and $j = 2$ (6-fold degenerate).
- The above results were first obtained by (Dasgupta, Sheikh-Jabbari, Raamsdonk, 2002) from the BMN matrix model point-of-view.
- In the flat-space limit ($\mu \to 0$), we recover the results of (Axenides-Floratos-Perivolaropoulos, 2000, 2001).

# Spherically symmetric membrane

# The spherically symmetric membrane

- Let us now consider the fully symmetric case:

$$u \equiv u_1 = u_2 = u_3 .$$

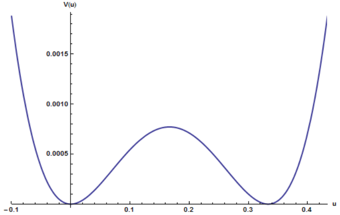
# The spherically symmetric membrane

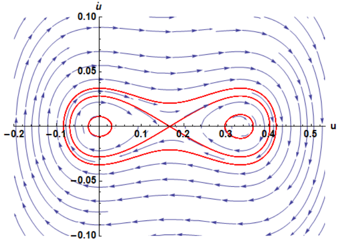
- Let us now consider the fully symmetric case:

$$u \equiv u_1 = u_2 = u_3.$$

- The membrane Hamiltonian is that of a double-well oscillator:

$$H = 2\pi T \mu^4 \left[ p^2 + u^2 \left( u - \frac{1}{3} \right)^2 \right].$$



# The spherically symmetric membrane

- Let us now consider the fully symmetric case:

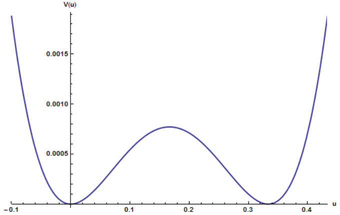
$$u \equiv u_1 = u_2 = u_3.$$

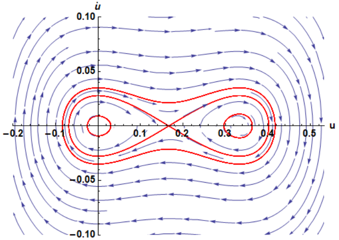
- The membrane Hamiltonian is that of a double-well oscillator:

$$H = 2\pi T \mu^4 \left[ p^2 + u^2 \left( u - \frac{1}{3} \right)^2 \right].$$

- Here are the corresponding equations of motion:

$$\dot{u} = p, \qquad \dot{p} = -u \left[ 2u^2 + \frac{1}{9} - u \right].$$



# The spherically symmetric membrane

- Let us now consider the fully symmetric case:

$$u \equiv u_1 = u_2 = u_3.$$

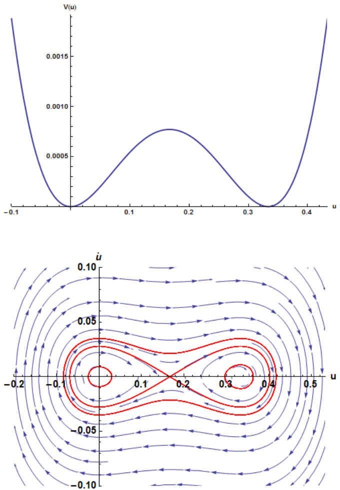
- The membrane Hamiltonian is that of a double-well oscillator:

$$H = 2\pi T \mu^4 \left[p^2 + u^2 \left(u - \frac{1}{3}\right)^2\right].$$

- Here are the corresponding equations of motion:

$$\dot{u} = p, \qquad \dot{p} = -u\left[2u^2 + \frac{1}{9} - u\right].$$

- Define $\mathcal{E} \equiv E/2\pi T\mu^4$, $\mathcal{E}_c \equiv 6^{-4}$. There are three kinds of orbits:



# The spherically symmetric membrane

- Let us now consider the fully symmetric case:

$$u \equiv u_1 = u_2 = u_3.$$

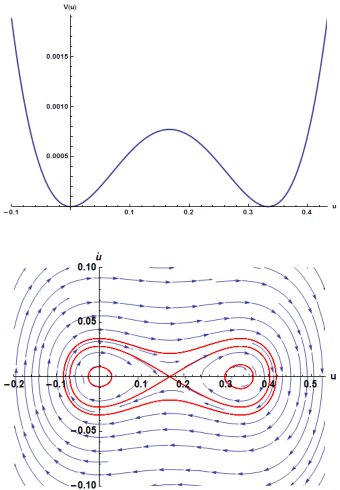
- The membrane Hamiltonian is that of a double-well oscillator:

$$H = 2\pi T\mu^4 \left[p^2 + u^2\left(u - \frac{1}{3}\right)^2\right].$$

- Here are the corresponding equations of motion:

$$\dot{u} = p, \qquad \dot{p} = -u\left[2u^2 + \frac{1}{9} - u\right].$$

- Define $\mathcal{E} \equiv E/2\pi T\mu^4$, $\mathcal{E}_c \equiv 6^{-4}$. There are three kinds of orbits:
  • (1) oscillations of small energies ($\mathcal{E} < \mathcal{E}_c$) around either of the two stable global minima ($u_0 = 0, 1/3$)



# The spherically symmetric membrane

- Let us now consider the fully symmetric case:

$$u \equiv u_1 = u_2 = u_3.$$

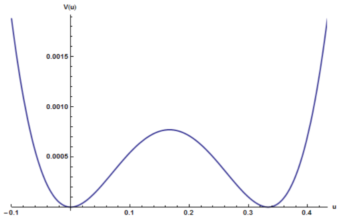
- The membrane Hamiltonian is that of a double-well oscillator:

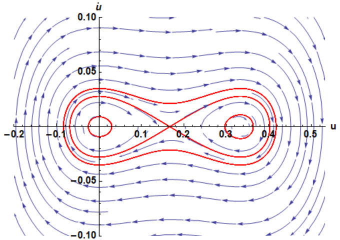
$$H = 2\pi T \mu^4 \left[ p^2 + u^2 \left( u - \frac{1}{3} \right)^2 \right].$$

- Here are the corresponding equations of motion:

$$\dot{u} = p, \qquad \dot{p} = -u \left[ 2u^2 + \frac{1}{9} - u \right].$$

- Define $\mathcal{E} \equiv E/2\pi T \mu^4$, $\mathcal{E}_c \equiv 6^{-4}$. There are three kinds of orbits: • (1) oscillations of small energies ($\mathcal{E} < \mathcal{E}_c$) around either of the two stable global minima ($u_0 = 0, 1/3$) • (2) oscillations of larger energies ($\mathcal{E} > \mathcal{E}_c$) around the local maximum ($u_0 = 1/6$)



# The spherically symmetric membrane

- Let us now consider the fully symmetric case:

$$u \equiv u_1 = u_2 = u_3.$$

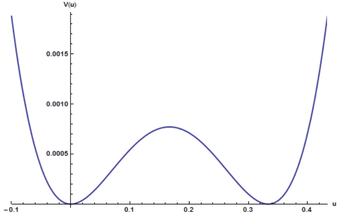
- The membrane Hamiltonian is that of a double-well oscillator:

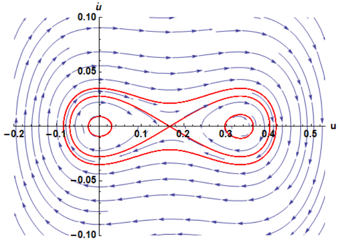
$$H = 2\pi T\mu^4 \left[p^2 + u^2 \left(u - \frac{1}{3}\right)^2\right].$$

- Here are the corresponding equations of motion:

$$\dot{u} = p, \qquad \dot{p} = -u\left[2u^2 + \frac{1}{9} - u\right].$$

- Define $\mathcal{E} \equiv E/2\pi T\mu^4$, $\mathcal{E}_c \equiv 6^{-4}$. There are three kinds of orbits: • (1) oscillations of small energies ($\mathcal{E} < \mathcal{E}_c$) around either of the two stable global minima ($u_0 = 0, 1/3$) • (2) oscillations of larger energies ($\mathcal{E} > \mathcal{E}_c$) around the local maximum ($u_0 = 1/6$) • (3) two homoclinic orbits through the unstable equilibrium point at $u_0 = 1/6$ with energy equal to the potential height ($\mathcal{E} = \mathcal{E}_c$).



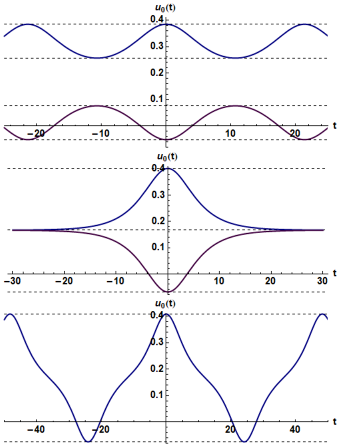



# The spherically symmetric membrane

- The orbits may be computed from the initial conditions:

$$\dot{u}_0\,(0) = 0, \qquad u_0\,(0) = \frac{1}{6} \pm \sqrt{\frac{1}{6^2} + \sqrt{\mathcal{E}}},$$

where the $\pm$ signs correspond to the right/left side of the double-well potential.



# The spherically symmetric membrane

- The orbits may be computed from the initial conditions:

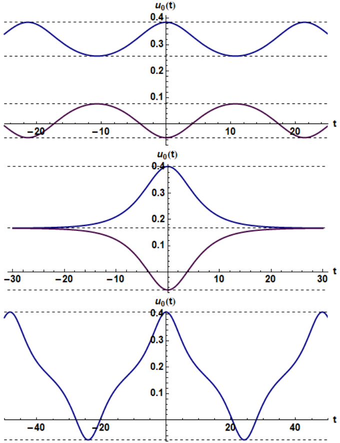
$$\dot{u}_0(0) = 0, \qquad u_0(0) = \frac{1}{6} \pm \sqrt{\frac{1}{6^2} + \sqrt{\mathcal{E}}},$$

where the $\pm$ signs correspond to the right/left side of the double-well potential.

- Integrating the energy integral we find the solution:

$$u_0(t) = \frac{1}{6} \pm \sqrt{\frac{1}{6^2} + \sqrt{\mathcal{E}}} \cdot cn\left[\sqrt{2\sqrt{\mathcal{E}}} \cdot t \left| \frac{1}{2}\left(1 + \frac{1}{36\sqrt{\mathcal{E}}}\right)\right.\right],$$

where only the plus sign should be kept for $\mathcal{E} \geq \mathcal{E}_c$.



# The spherically symmetric membrane

- The orbits may be computed from the initial conditions:

$$\dot{u}_0(0) = 0, \qquad u_0(0) = \frac{1}{6} \pm \sqrt{\frac{1}{6^2} + \sqrt{\mathcal{E}}},$$

where the $\pm$ signs correspond to the right/left side of the double-well potential.

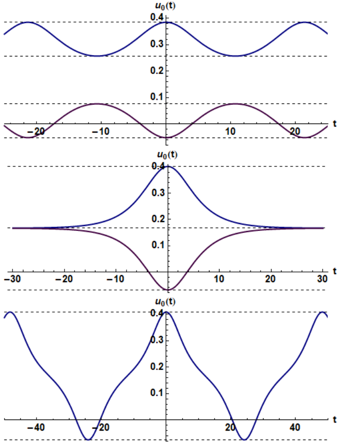
- Integrating the energy integral we find the solution:

$$u_0(t) = \frac{1}{6} \pm \sqrt{\frac{1}{6^2} + \sqrt{\mathcal{E}}} \cdot cn\left[\sqrt{2\sqrt{\mathcal{E}}} \cdot t \middle| \frac{1}{2}\left(1 + \frac{1}{36\sqrt{\mathcal{E}}}\right)\right],$$

where only the plus sign should be kept for $\mathcal{E} \geq \mathcal{E}_c$.

- For the critical energy $\mathcal{E} = \mathcal{E}_c$ the homoclinic orbit is obtained:

$$u_0(t) = \frac{1}{6} \pm \frac{1}{3\sqrt{2}} \cdot \text{sech}\left(\frac{t}{3\sqrt{2}}\right).$$

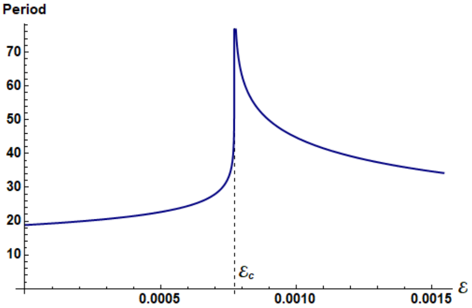


# The spherically symmetric membrane

- The period as a function of the energy is given in terms of the complete elliptic integral of the first kind:

$$T(\mathcal{E}) = 2\sqrt{\frac{2}{\sqrt{\mathcal{E}}}} \cdot \mathbf{K}\left(\frac{1}{2}\left(1 + \frac{1}{36\sqrt{\mathcal{E}}}\right)\right),$$

it becomes infinite for the homoclinic orbit $\mathcal{E} = \mathcal{E}_c$. For more, see e.g. Brizard-Westland (2017).



# Conclusions & outlook

## Outlook

- Radial & angular perturbation analysis in the $SO(3) \times SO(6)$ case in (Axenides-Floratos-G.L., 2017, 2018).

# Outlook

- Radial & angular perturbation analysis in the $SO(3) \times SO(6)$ case in (Axenides-Floratos-G.L., 2017, 2018).
- Analysis of higher orders in perturbation theory... instability cascade... (Axenides-Floratos-G.L., to appear).

# Outlook

- Radial & angular perturbation analysis in the $SO(3) \times SO(6)$ case in (Axenides-Floratos-G.L., 2017, 2018).
- Analysis of higher orders in perturbation theory... instability cascade... (Axenides-Floratos-G.L., to appear).
- Study of chaotic dynamics... KAM theory...

# Outlook

- Radial & angular perturbation analysis in the $SO(3) \times SO(6)$ case in (Axenides-Floratos-G.L., 2017, 2018).
- Analysis of higher orders in perturbation theory... instability cascade... (Axenides-Floratos-G.L., to appear).
- Study of chaotic dynamics... KAM theory...
- Generalization to M5-branes...

# Outlook

- Radial & angular perturbation analysis in the $SO(3) \times SO(6)$ case in (Axenides-Floratos-G.L., 2017, 2018).
- Analysis of higher orders in perturbation theory... instability cascade... (Axenides-Floratos-G.L., to appear).
- Study of chaotic dynamics... KAM theory...
- Generalization to M5-branes...
- Uplift to the matrix model...

# Outlook

- Radial & angular perturbation analysis in the $SO(3) \times SO(6)$ case in (Axenides-Floratos-G.L., 2017, 2018).
- Analysis of higher orders in perturbation theory... instability cascade... (Axenides-Floratos-G.L., to appear).
- Study of chaotic dynamics... KAM theory...
- Generalization to M5-branes...
- Uplift to the matrix model...

Ευχαριστώ!

# Plane-wave backgrounds

# Plane-fronted gravitational waves with parallel rays (pp-waves)

- pp-waves were originally introduced in the so-called Brinkmann coordinates:

$$ds^2 = H(u,x,y)du^2 + 2dudv + dx^2 + dy^2.$$

Brinkmann (1925)

- Equivalently, they can be defined as spacetimes that admit a covariantly constant null Killing vector:

$$\nabla_m k_n = 0, \qquad k^n k_n = 0.$$

Ehlers-Kundt (1962)

- Plane-wave spacetimes are special pp-waves; the gravitational wave analogue of e/m plane waves...

# pp-waves & plane-waves in $d+1$ dimensions

- The metric of a pp-wave in $d+1$ dimensions is given by:

$$ds^2 = -2dx^+dx^- - F(x^+, x^i)dx^+dx^+ + 2A_j(x^+, x^i)dx^+dx^j + g_{jk}(x^+, x^i)dx^j dx^k, \quad x^\pm \equiv \frac{1}{\sqrt{2}}\left(x^0 \pm x^d\right),$$

where $i, j = 1, 2, \ldots d-1$ and $(F, A_j, g_{jk})$ are determined from the supergravity equations of motion.

# pp-waves & plane-waves in $d+1$ dimensions

- The metric of a pp-wave in $d+1$ dimensions is given by:

  $$ds^2 = -2dx^+dx^- - F(x^+,x^i)dx^+dx^+ + 2A_j(x^+,x^i)dx^+dx^j + g_{jk}(x^+,x^i)dx^jdx^k, \quad x^\pm \equiv \frac{1}{\sqrt{2}}\left(x^0 \pm x^d\right),$$

  where $i,j = 1,2,\dots d-1$ and $(F, A_j, g_{jk})$ are determined from the supergravity equations of motion.

- For $A_j = 0$, $g_{jk} = \delta_{jk}$, pp-waves are $\alpha'$–exact supergravity solutions:

  $$ds^2 = -2dx^+dx^- - F(x^+,x^i)dx^+dx^+ + dx^idx^i.$$

# pp-waves & plane-waves in $d+1$ dimensions

- The metric of a pp-wave in $d+1$ dimensions is given by:

$$ds^2 = -2dx^+dx^- - F(x^+, x^i)dx^+dx^+ + 2A_j(x^+, x^i)dx^+dx^j + g_{jk}(x^+, x^i)dx^j dx^k, \quad x^{\pm} \equiv \frac{1}{\sqrt{2}}\left(x^0 \pm x^d\right),$$

where $i, j = 1, 2, \ldots d-1$ and $(F, A_j, g_{jk})$ are determined from the supergravity equations of motion.

- For $A_j = 0$, $g_{jk} = \delta_{jk}$, pp-waves are $\alpha'$–exact supergravity solutions:

$$ds^2 = -2dx^+dx^- - F(x^+, x^i)dx^+dx^+ + dx^i dx^i.$$

- Plane-waves are pp-waves for which $F(x^+, x^i) = f_{ij}(x^+)x^i x^j$, $A_j = 0$ and $g_{jk} = \delta_{jk}$:

$$ds^2 = -2dx^+dx^- - f_{ij}(x^+)x^i x^j dx^+dx^+ + dx^i dx^i.$$

# pp-waves & plane-waves in $d+1$ dimensions

- The metric of a pp-wave in $d+1$ dimensions is given by:

$$ds^2 = -2dx^+dx^- - F(x^+,x^i)dx^+dx^+ + 2A_j(x^+,x^i)dx^+dx^j + g_{jk}(x^+,x^i)dx^jdx^k, \quad x^\pm \equiv \frac{1}{\sqrt{2}}\left(x^0 \pm x^d\right),$$

where $i,j = 1,2,\ldots d-1$ and $(F, A_j, g_{jk})$ are determined from the supergravity equations of motion.

- For $A_j = 0$, $g_{jk} = \delta_{jk}$, pp-waves are $\alpha'$–exact supergravity solutions:

$$ds^2 = -2dx^+dx^- - F(x^+,x^i)dx^+dx^+ + dx^idx^i.$$

- Plane-waves are pp-waves for which $F(x^+,x^i) = f_{ij}(x^+)x^ix^j$, $A_j = 0$ and $g_{jk} = \delta_{jk}$:

$$ds^2 = -2dx^+dx^- - f_{ij}(x^+)x^ix^jdx^+dx^+ + dx^idx^i.$$

- Homogeneous plane-waves have $f_{ij}(x^+) = \mu_{ij}^2$, constant:

$$ds^2 = -2dx^+dx^- - \mu_{ij}^2x^ix^jdx^+dx^+ + dx^idx^i.$$

# pp-waves & plane-waves in $d+1$ dimensions

- The metric of a pp-wave in $d+1$ dimensions is given by:

$$ds^2 = -2dx^+dx^- - F(x^+, x^i)dx^+dx^+ + 2A_j(x^+, x^i)dx^+dx^j + g_{jk}(x^+, x^i)dx^j dx^k, \quad x^\pm \equiv \frac{1}{\sqrt{2}}\left(x^0 \pm x^d\right),$$

  where $i, j = 1, 2, \ldots d-1$ and $(F, A_j, g_{jk})$ are determined from the supergravity equations of motion.

- For $A_j = 0$, $g_{jk} = \delta_{jk}$, pp-waves are $\alpha'$–exact supergravity solutions:

$$ds^2 = -2dx^+dx^- - F(x^+, x^i)dx^+dx^+ + dx^i dx^i.$$

- Plane-waves are pp-waves for which $F(x^+, x^i) = f_{ij}(x^+)x^i x^j$, $A_j = 0$ and $g_{jk} = \delta_{jk}$:

$$ds^2 = -2dx^+dx^- - f_{ij}(x^+)x^i x^j dx^+dx^+ + dx^i dx^i.$$

- Homogeneous plane-waves have $f_{ij}(x^+) = \mu_{ij}^2$, constant:

$$ds^2 = -2dx^+dx^- - \mu_{ij}^2 x^i x^j dx^+dx^+ + dx^i dx^i.$$

- Homogeneous and isotropic plane-waves have $\mu_{ij} = \mu$:

$$ds^2 = -2dx^+dx^- - \mu^2 x^i x^i dx^+dx^+ + dx^i dx^i.$$

# Properties of plane waves

- Penrose limit: Any spacetime has a plane-wave as a limit (Penrose, 1975 & Güven, 2000).
- $\alpha'$-exact solutions of supergravity (with or without flux terms).
  Amati-Klimčík (1988), Horowitz-Steif (1990)
- Maximally susy backgrounds of 11-dimensional & IIB sugra (along with flat space and $\mathrm{AdS}_{4/5/7} \times \mathrm{S}^{7/5/4}$).
  Figueroa-O'Farrill & Papadopoulos (2003)
- IIB superstring $\sigma$ model exactly solvable & quantizable on the 10-dimensional maximally susy background.
  Metsaev (2001), Metsaev-Tseytlin (2002)
- BMN sector of $\mathrm{AdS}_5/\mathrm{CFT}_4$: Penrose limit of IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5 \leftrightarrow$ BMN limit of $\mathcal{N} = 4$ SYM
  Berenstein-Maldacena-Nastase (2002)

# Membranes in the light-cone gauge

# Bosonic membrane in a curved background

- Dirac-Nambu-Goto (DNG) action:

$$\mathcal{S}_{\text{DNG}} = -T \int d\tau d^2\sigma \left\{ \sqrt{-h} + \dot{X}^m \partial_1 X^n \partial_2 X^r A_{rnm}(X) \right\}, \qquad T \equiv \frac{1}{(2\pi)^2 \ell_{11}^3},$$

where $(m, n, r, s = 0, \ldots, 10)$,

$$h_{ij} \equiv G_{mn} \partial_i X^m \partial_j X^n \quad \text{(induced metric)} \quad h \equiv \det h_{ij} \qquad \& \qquad F_{mnrs} = 4\partial_{[m} A_{nrs]} \quad \text{(field strength)}$$

and $A_{nrs}$ is the (antisymmetric) 3-form field of 11-dimensional supergravity.

# The light-cone gauge

- In the light-cone gauge, we write:

$$X^{\pm} = \frac{1}{\sqrt{2}}\left(X^0 \pm X^{10}\right) \qquad \& \qquad X^+ = \tau.$$

Goldstone-Hoppe (1982)

- The light-cone Hamiltonian is then written as follows ($G_{--} = G_{a-} = 0$):

$$H = T\int d^2\sigma \Bigg\{ \frac{1}{2}\frac{G_{+-}}{P_- - C_-}\left[\left(P_a - C_a - \frac{P_- - C_-}{G_{+-}}G_{a+}\right)^2 + \frac{1}{2}G_{ab}G_{cd}\{X^a, X^c\}\{X^b, X^d\}\right] -$$

$$-\frac{1}{2}\frac{P_- - C_-}{G_{+-}}G_{++} - C_+ + \frac{1}{P_- - C_-}\left[C_- C_{+-} - \{X^a, X^b\}P_a C_{+-b}\right]\Bigg\},$$

de Wit, Peeters, Plefka (1998)

where $(a, b, c, d = 1, \ldots, 9)$,

$$C_\pm \equiv C_{\pm ab} - \partial_1 X^a \partial_2 X^b, \quad C_{+-} \equiv -C_{+-a}\{X^-, X^a\}, \quad C_a \equiv -\left(C_{-ab}\{X^b, X^-\} + C_{abc}\partial_1 X^b \partial_2 X^c\right).$$

# Poisson bracket

The Poisson bracket is defined as:

$$\{f\,,\,g\} \equiv \frac{\epsilon_{rs}}{\sqrt{w\left(\boldsymbol{\sigma}\right)}}\,\partial_r f\;\partial_s g = \frac{1}{\sqrt{w\left(\boldsymbol{\sigma}\right)}}\left(\partial_1 f\;\partial_2 g - \partial_2 f\;\partial_1 g\right),$$

where $d^2\sigma = \sqrt{w\left(\boldsymbol{\sigma}\right)}\;d\sigma_1\;d\sigma_2$. In a flat space-sheet, $w\left(\boldsymbol{\sigma}\right) = 1$.

# Poisson bracket

The Poisson bracket is defined as:

$$\{f\,,\,g\} \equiv \frac{\epsilon_{rs}}{\sqrt{w\left(\boldsymbol{\sigma}\right)}}\,\partial_r f\;\partial_s g = \frac{1}{\sqrt{w\left(\boldsymbol{\sigma}\right)}}\left(\partial_1 f\;\partial_2 g - \partial_2 f\;\partial_1 g\right),$$

where $d^2\sigma = \sqrt{w\left(\boldsymbol{\sigma}\right)}\;d\sigma_1\;d\sigma_2$. In a flat space-sheet, $w\left(\boldsymbol{\sigma}\right) = 1$. In spherical coordinates, $\sqrt{w\left(\theta\right)} = \sin\theta$:

$$(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta), \qquad \phi\in[0,2\pi), \quad \theta\in[0,\pi].$$

# Light-cone gauge in the plane-wave background

- In the maximally supersymmetric plane-wave background,

$$G_{+-} = -1, \qquad G_{ab} = \delta_{ab}, \qquad G_{++} = -\frac{\mu^2}{9}\sum_{i=1}^{3} x^i x^i - \frac{\mu^2}{36}\sum_{j=1}^{6} y^j y^j, \qquad G_{--} = G_{a\pm} = 0$$

$$C_{-} = C_{+-} = C_a = 0, \qquad C_{+} = \frac{\mu}{3}\epsilon_{ijk}\partial_1 x^i \partial_2 x^j x^k,$$

the light-cone Hamiltonian becomes (for $P_{-} = -1$):

$$H = \frac{T}{2}\int d^2\sigma \left[p^2 + \frac{1}{2}\left\{x^i, x^j\right\}^2 + \frac{1}{2}\left\{y^i, y^j\right\}^2 + \left\{x^i, y^j\right\}^2 + \frac{\mu^2 x^2}{9} + \frac{\mu^2 y^2}{36} - \frac{\mu}{3}\epsilon_{ijk}\left\{x^i, x^j\right\} x^k\right].$$

- The corresponding equations of motion and the Gauß law constraint read:

$$\ddot{x}_i = \left\{\left\{x_i, x_j\right\}, x_j\right\} + \left\{\left\{x_i, y_j\right\}, y_j\right\} - \frac{\mu^2}{9} x_i + \frac{\mu}{2}\epsilon_{ijk}\left\{x_j, x_k\right\}, \qquad \sum_{i=1}^{3}\left\{\dot{x}^i, x^i\right\} + \sum_{j=1}^{6}\left\{\dot{y}^j, y^j\right\} = 0$$

$$\ddot{y}_i = \left\{\left\{y_i, y_j\right\}, y_j\right\} + \left\{\left\{y_i, x_j\right\}, x_j\right\} - \frac{\mu^2}{36} y_i.$$

# Matrix models

# M-theory on a plane wave

- Much of our interest in plane-wave backgrounds derives from the fact that the BMN matrix model,

$$H = \mathrm{Tr}\left[\frac{1}{2}\dot{\mathbf{X}}^2 - \frac{1}{4}\left[\mathbf{X}^i, \mathbf{X}^j\right]^2\right] + \frac{1}{2}\cdot\mathrm{Tr}\left[\sum_{i=1}^{3}\frac{\mu^2}{9}\mathbf{X}_i^2 + \sum_{j=4}^{9}\frac{\mu^2}{36}\mathbf{X}_j^2 + \frac{2i\mu}{3}\epsilon_{ijk}\mathbf{X}_i\mathbf{X}_j\mathbf{X}_k\right] + \text{fermions},$$

Berenstein-Maldacena-Nastase (2002)

describes M-Theory on the 11-dimensional maximally supersymmetric plane-wave background:

$$ds^2 = -2dx^+dx^- - \left[\frac{\mu^2}{9}\sum_{i=1}^{3}x^ix^i + \frac{\mu^2}{36}\sum_{j=1}^{6}y^jy^j\right]dx^+dx^+ + dx^idx^i + dy^jdy^j, \qquad F_{123+} = \mu.$$

# M-theory on a plane wave

- Much of our interest in plane-wave backgrounds derives from the fact that the BMN matrix model,

$$H = \mathrm{Tr}\left[\frac{1}{2}\dot{\mathbf{X}}^2 - \frac{1}{4}\left[\mathbf{X}^i, \mathbf{X}^j\right]^2\right] + \frac{1}{2}\cdot\mathrm{Tr}\left[\sum_{i=1}^{3}\frac{\mu^2}{9}\mathbf{X}_i^2 + \sum_{j=4}^{9}\frac{\mu^2}{36}\mathbf{X}_j^2 + \frac{2i\mu}{3}\epsilon_{ijk}\mathbf{X}_i\mathbf{X}_j\mathbf{X}_k\right] + \text{fermions},$$

Berenstein-Maldacena-Nastase (2002)

describes M-Theory on the 11-dimensional maximally supersymmetric plane-wave background:

$$ds^2 = -2dx^+dx^- - \left[\frac{\mu^2}{9}\sum_{i=1}^{3}x^ix^i + \frac{\mu^2}{36}\sum_{j=1}^{6}y^jy^j\right]dx^+dx^+ + dx^idx^i + dy^jdy^j, \qquad F_{123+} = \mu.$$

- Here's the full Hamiltonian of the BMN matrix model (with the fermions):

$$H = H_0 + \frac{1}{2}\cdot\mathrm{Tr}\left[\sum_{i=1}^{3}\frac{\mu^2}{9}\mathbf{X}_i^2 + \sum_{j=4}^{9}\frac{\mu^2}{36}\mathbf{X}_j^2 - \frac{i\mu}{8}\theta^T\gamma_{123}\,\theta + \frac{2i\mu}{3}\epsilon_{ijk}\mathbf{X}_i\mathbf{X}_j\mathbf{X}_k\right],$$

where $H_0$ is just the BFSS Hamiltonian describing M-theory in flat ($\mu = 0$) space:

$$H_0 = \mathrm{Tr}\left[\frac{1}{2}\dot{\mathbf{X}}^2 - \frac{1}{4}\left[\mathbf{X}^i, \mathbf{X}^j\right]^2 + \theta^T\gamma_i\left[\mathbf{X}^i, \theta\right]\right] \qquad \text{(Banks-Fischler-Shenker-Susskind, 1996).}$$

# M-theory on a plane wave from membranes

- As shown by Dasgupta, Sheikh-Jabbari and Van Raamsdonk in 2002, the BMN matrix model can be derived by regularizing the light-cone (super)membrane in the 11-dimensional maximally susy plane-wave background.

# M-theory on a plane wave from membranes

- As shown by Dasgupta, Sheikh-Jabbari and Van Raamsdonk in 2002, the BMN matrix model can be derived by regularizing the light-cone (super)membrane in the 11-dimensional maximally susy plane-wave background.
- Equivalently, the light-cone (super)membrane on the maximally susy plane-wave background can be seen as the continuum/classical/large-$N$ limit (matrix dimensionality $N \to \infty$) of the BMN matrix model.

# M-theory on a plane wave from membranes

- As shown by Dasgupta, Sheikh-Jabbari and Van Raamsdonk in 2002, the BMN matrix model can be derived by regularizing the light-cone (super)membrane in the 11-dimensional maximally susy plane-wave background.

- Equivalently, the light-cone (super)membrane on the maximally susy plane-wave background can be seen as the continuum/classical/large-$N$ limit (matrix dimensionality $N \to \infty$) of the BMN matrix model.