Stability of nonlinear waves: theory and examples

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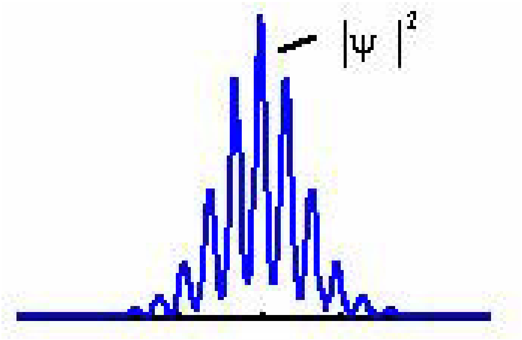
Outline of the talk

- G. Mylonas (Nice), M. Agaoglou (Comenius University)
- P.G.Kevrekidis (UMass), D. Frantzeskakis (UoA)
- H. Sussanto (Essex), M. Feckan (Comenius University)
- A. Vakakis (Illinois), D. Pelinovsky (McMaster)

Discrete Solitons

Discrete solitons were first suggested by Davydov in alpha-helix proteins. This model attempted to explain some fundamental issues in biophysics such as for example storage of phonon energy in proteins.

$i\hbar \frac{d\Psi_n}{dt} + J(\Psi_{n+1} - \Psi_{n-1}) + \sigma |\Psi_n|^2 \Psi_n = 0$



Integrable vs Non-integrable lattices

sine-Gordon (Integrable) VS Discrete SG (non-integrable)

$$u_{tt} - u_{xx} = \Gamma \sin u, \quad \text{vs} \quad \ddot{u}_n = u_{n+1} - 2u_n + u_{n-1} + \Gamma \sin u_n$$

NSL (intgrable) VS Ablowitz-Ladik lattice (Integrable)/DNLS (non-integrable)

$$\mathrm{i}\, u_t = 2|u|^2 u + u_{xx}$$

VS

$$\mathrm{i}\, \dot{u}_n = |u_n|^2 (u_{n+1} + u_{n-1}) + \frac{1}{h^2}(u_{n+1} - 2u_n + u_{n-1})$$

and

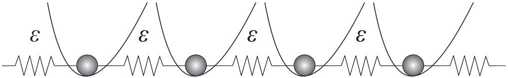
$$\mathrm{i}\, \dot{u}_n = 2u_n |u_n|^2 + \frac{1}{h^2}(u_{n+1} - 2u_n + u_{n-1})$$

Klein-Gordon Lattice

Klein-Gordon (KG) lattice models a chain of coupled anharmonic oscillators with nearest-neighbour interactions

$$\ddot{u}_n + V'(u_n) = \varepsilon(u_{n-1} - 2u_n + u_{n+1}).$$

where $\{u_n(t)\}_{n\in\mathbb{Z}} : \mathbb{R} \to \mathbb{R}^{\mathbb{Z}}$, dot represents time derivative, ε is the coupling constant, and $V : \mathbb{R} \to \mathbb{R}$ is an on-site potential.



Applications:

- dislocations in crystals (e.g. Frenkel & Kontorova 1938)
- oscillations in biological molecules (e.g. Peyrard & Bishop 1989)

Anharmonic oscillator

We make the following assumptions:

- $V(u) = u^2 \pm u^4 + \mathcal{O}(u^5)$, where $+/$ corresponds to hard/soft potential;
- $0 < \varepsilon \ll 1$: oscillators are weakly coupled.

In the anti-continuum limit ($\varepsilon = 0$), each oscillator is governed by

$$\ddot{\varphi} + V'(\varphi) = 0, \quad \Rightarrow \quad \frac{1}{2}\dot{\varphi}^2 + V(\varphi) = E,$$

where $\varphi \in H^2_{per}(0, T)$. Period of oscillations T is uniquely defined by the energy level E, according to the following formula:

$$T = \sqrt{2} \int_{-a}^{a} \frac{d\varphi}{\sqrt{E - V(\varphi)}}.$$

Multi-breathers in the anti-continuum limit

Breathers are spatially localized time-periodic solutions to the Klein-Gordon lattice. Multi-breathers are constructed by parameter continuation in ε from $\varepsilon = 0$. For $\varepsilon = 0$ we take

$$\mathbf{u}^{(0)}(t) = \sum_{k \in S} \sigma_k \varphi(t) \mathbf{e}_k \quad \in \quad l^2\left(\mathbb{Z}, H^2_{per}(0, T)\right),$$

where $S \subset \mathbb{Z}$ is the set of excited sites and $\mathbf{e}_k$ is the unit vector in $l^2(\mathbb{Z})$ at the node k. The oscillators are in phase if $\sigma_k = +1$ and out-of-phase if $\sigma_k = 1$.

Theorem (MacKay & Aubry 1994)

Fix the period $T \neq 2\pi n$, $n \in \mathbb{N}$ and the T-periodic solution $\varphi \in H^2_{per}(0, T)$ of the anharmonic oscillator equation for $T(E) \neq 0$. There exist $\varepsilon_0 > 0$ and $C > 0$ such that $\forall \varepsilon(\varepsilon_0, \varepsilon_0)$ there exists a solution $\mathbf{u}^{(\epsilon)} \in l^2\left(\mathbb{Z}, H^2_{per}(0, T)\right)$, of the Klein-Gordon lattice satisfying

$$\left\| \mathbf{u}^{(\epsilon)} - \mathbf{u}^{(0)} \right\|_{l^2\left(\mathbb{Z}, H^2_{per}(0,T)\right)} \leq C\epsilon$$

Floquet Multipliers

Linearize about the breather solution to the dKG by replacing **u** with **u**+**w**, where $\mathbf{w} : \mathbb{R} \to \mathbb{R}^{\mathbb{Z}}$ is a small perturbation, and collect the terms linear in **w**:

$$\ddot{\mathbf{w}}_n + V''(\mathbf{u}_n)\mathbf{w}_n = \epsilon\left(\mathbf{w}_{n+1} - 2\mathbf{w}_n + \mathbf{w}_{n-1}\right), \qquad n \in \mathbb{Z}$$

In the *anti-continuum limit*, it is easy to find the Floquet multipliers:

- on "holes" $n \in \mathbb{Z} \setminus S$,

$$\ddot{\mathbf{w}}_n + \mathbf{w}_n = 0, \qquad \begin{pmatrix} \mathbf{w}_n(T) \\ \dot{\mathbf{w}}_n(T) \end{pmatrix} = \begin{pmatrix} \cos T & \sin T \\ -\sin T & \cos T \end{pmatrix} \begin{pmatrix} \mathbf{w}_n(0) \\ \dot{\mathbf{w}}_n(0) \end{pmatrix},$$

  Floquet multipliers are $\mu_{1,2} = \mathrm{e}^{\pm iT}$

- on excited sites, $n \in S$,

$$\ddot{\mathbf{w}}_n + V''(\varphi)\mathbf{w}_n = 0, \qquad \begin{pmatrix} \mathbf{w}_n(T) \\ \dot{\mathbf{w}}_n(T) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ T'(E)(V'(a))^2 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{w}_n(0) \\ \dot{\mathbf{w}}_n(0) \end{pmatrix},$$

  Floquet multipliers are $\mu_{1,2} = 1$ of geometric multiplicity 1 and algebraic multiplicity 2.

Floquet Exponent

A Floquet multiplier μ can be written as $\mu = e^{\lambda T}$

Theorem (Pelinovsky '12)

For small $\epsilon > 0$ the linearized stability problem has $2M$ small Floquet exponents $\lambda = \epsilon^{N/2}\Lambda + O\left(\epsilon^{(N+1)/2}\right)$, where λ is determined from the eigenvalue problem

$$-\frac{T(E)^2}{2T'(E)K_N}\Lambda^2 \boldsymbol{c} = S\boldsymbol{c}, \qquad \boldsymbol{c} \in \mathbb{C}^M$$

with $S \in \mathbb{R}^{M\times M}$ is a tridiagonal matrix with elements

$$S_{i,j} = -\sigma_j\left(\sigma_{j-1} + \sigma_{j+1}\right)\delta_{i,j} + \delta_{i,j-1} + \delta_{i,j+1}, \qquad 1 \le i,j \le M,$$

and K_N is defined by

$$K_N = \int_0^T \dot{\varphi}(t)\dot{\varphi}_{N-1}(t)dt, \qquad \left(\partial_t^2 + 1\right)\varphi_k = \varphi_{k-1}, \quad \varphi_0 = \varphi.$$

Stability of Multi-breathers

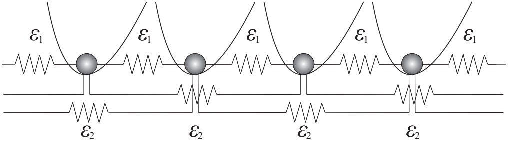
Sandstede (1998) showed that the matrix S has exactly n_0 positive and $M-1-n_0$ negative eigenvalues in addition to the simple zero eigenvalue, where $n_0 =$(sign changes in n). Hence, stability of multibreathers is determined by the sign $T'(E)K_N(T)$ and the phase parameters $\{\sigma_k\}_{k=1}^{M-1}$.

Theorem

If $T'(E)K_N(T) > 0$ the linearized problem for the multibreathers has exactly n_0 pairs of "stable" Floquet exponents and $M-1-n_0$ pairs of "unstable" Floquet exponents counting their multiplicities. If $T'(E)K_N(T) < 0$ the conclusion changes to the opposite.

KG with LRI

The picture *radically* changes when the chain involves interactions with range longer than mere nearest neighbors. The range parameter r will be used to indicate the interaction length between the oscillators of the chain. The next nearest neighbor (NNN) chain the range is $r = 2$. The coupling force between the oscillators of the chain is linear and the coupling constants $\varepsilon_i, i = 1 \dots r$ are not, in general, equal.



The Hamiltonian of a 1D KG chain with long range interactions is:

$$H = \sum_{i=-\infty}^{\infty} [\frac{p_i^2}{2} + V(x_i)] + \frac{1}{2} \sum_{i=-\infty}^{\infty} \sum_{j=1}^{r} \varepsilon_j (x_i - x_{i+j})^2 \tag{4.1}$$

which leads to the equations of motion

$$\ddot{x}_i = -V'(x_i) + \sum_{j=1}^{r} \varepsilon_j (x_{i-j} - 2x_i + x_{i+j})$$

3-site breathers with $r = 2$

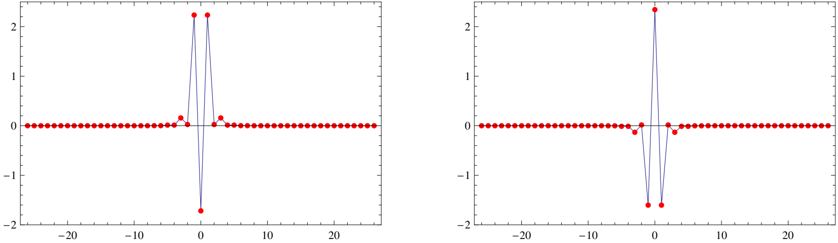


Figure: [Color online] Two snapshots of a 3-site ($n = 2$), anti-phase ($\phi_1 = \phi_2 = \pi$) multibreather in a range $r = 2$ Klein-Gordon chain with $\varepsilon_1 = \varepsilon_2 = 0.02$ and frequency $\mathrm{w} = 2\pi/7$.

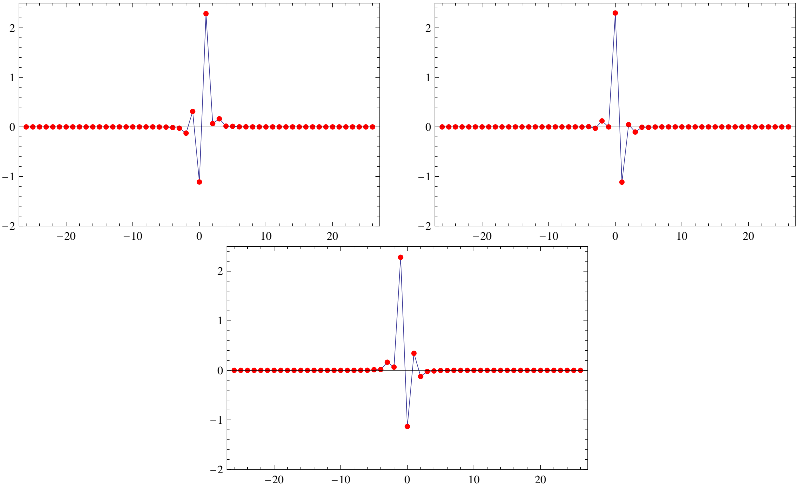
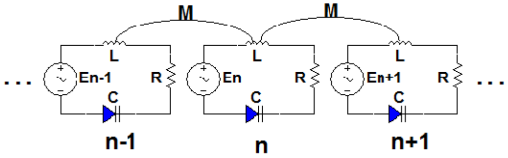


Figure: [Color online] Three snapshots of a 3-site ($n = 2$), phase-shift ($\phi_1 = \phi_2 \neq 0, \pi$) multibreather in a range $r = 2$ Klein-Gordon chain with $\varepsilon_1 = \varepsilon_2 = 0.02$ and frequency $\mathrm{w} = 2\pi/7$.

Nonlinear Magnetic Metamaterials

Lazarides *et al* PRL97 (2006) studied the discrete breathers in Nonlinear Megnetic Metamaterials, described by a dissipative lattice equation.
They consider a planar 1D array of N identical split-ring resonator (SSR) with their axes prependicular to the plane. The dynamics of charge q_n and the current i_n circulating in the nth SSR is described by



$$\ddot{q}_n + V'(q_n) = \epsilon(\ddot{q}_{n+1} + \ddot{q}_{n-1}) + \gamma\dot{q}_n - f(t) \tag{5.1}$$

with loss coefficient γ, coupling parameter ϵ, external forcing f.
We investigate the existence of small periodic solutions of system (7.5) with and without periodic forcing, and the bifurcation of periodic solutions of (7.5) with small γ, λ and f.

The NMM lattice can be written as

$$\ddot{q}_n + V'(q_n) = \epsilon(\ddot{q}_{n+1} + \ddot{q}_{n-1}), \quad n \in \mathbb{Z}, \tag{5.2}$$

where $t \in \mathbb{R}$ is the evolution time, $q_n(t) \in \mathbb{R}$ is the normalized charge stored in the capacitor of the n-th split-ring resonator, $V : \mathbb{R} \to \mathbb{R}$ is a smooth on-site potential for the voltage across the slit of the n-th resonator, and $\epsilon > 0$ is the coupling constant from the mutual inductance. In particular, the voltage $u = f(q) = V'(q)$ is found by inverting the charge-voltage dependence near small charge:

$$q = u + \alpha u^3 \quad \Rightarrow \quad u = f(q) = q - \alpha q^3 + \mathcal{O}(q^5) \quad \text{as} \quad q \to 0, \tag{5.3}$$

where self-focusing ($\alpha > 0$) or self-defocusing ($\alpha < 0$) nonlinearity, correspond to the soft and hard potentials V respectively, for sufficiently small values of q. Note that $V(-q) = V(q)$ for the potential defined by (5.3).

Focus on

- consider spectral stability of multi-site discrete breathers in the limit of small coupling constant ϵ. This limit is referred usually as the *anti-continuum* limit.
- Existence of periodic travelling wave solutions and Bifurcation results for periodic travelling waves for perturbed NNM lattice.

Formalism

We set up the NMM lattice as an evolution in t in the phase space $C^2([0,T), l^2(\mathbb{Z}))$, $T > 0$ is the maximal existence time (which may be infinite).
Let us define the bounded operator

$$M(\epsilon) = I - \epsilon(\sigma_+ + \sigma_-) : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}),$$

where the shift operators $\sigma_\pm : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ are defined by

$$(\sigma_\pm \mathbf{q})_n = q_{n\pm1}, \quad n \in \mathbb{Z}. \tag{6.1}$$

For any $\epsilon \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, the operator $M(\epsilon) : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ is diagonally dominant and hence invertible and the inverse operator $M^{-1}(\epsilon) : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ is bounded. Moreover, the operator $M^{-1}(\epsilon)$ is analytic at $\epsilon = 0$ and admits the Taylor series,

$$M^{-1}(\epsilon) = I + \sum_{k=1}^{\infty} \epsilon^k (\sigma_+ + \sigma_-)^k, \quad \epsilon \in \left(-\frac{1}{2}, \frac{1}{2}\right). \tag{6.2}$$

As a result, the evolution problem of the discrete Klein–Gordon equation can be formulated in the abstract form

$$\frac{d^2\mathbf{q}}{dt^2} + M^{-1}(\epsilon)\mathbf{f}(\mathbf{q}) = \mathbf{0}, \tag{6.3}$$

where $(\mathbf{f}(\mathbf{q}))_n = V'(q_n)$.

Theorem (Rothos & Pelinosky, '14)

Let $V \in C^2(\mathbb{R})$ *and* $\mathbf{q}_0, \mathbf{q}_1 \in l^2(\mathbb{Z})$. *For any* $\epsilon \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, *there exist* $T > 0$ *and a local solution of the evolution problem (6.3) in the phase space* $\mathbf{q} \in C^2([0, T), l^2(\mathbb{Z}))$ *such that* $\mathbf{q}(0) = \mathbf{q}_0$ *and* $\dot{\mathbf{q}}(0) = \mathbf{q}_1$.

Multi-breathers are constructed by parameter continuation in ε from $\varepsilon = 0$. For $\varepsilon = 0$ we take

$$\mathbf{Q}^{(0)}(t) = \sum_{k \in S} \sigma_k \varphi(t) \mathbf{e}_k \quad \in \quad l^2 \left(\mathbb{Z}, H^2_{per}(0,T) \right), \tag{6.4}$$

where $S \subset \mathbb{Z}$ is the set of excited sites and $\mathbf{e}_k$ is the unit vector in $l^2(\mathbb{Z})$ at the node k. The oscillators are in phase if $\sigma_k = +1$ and out-of-phase if $\sigma_k = 1$.
In the anti-continuum limit ($\varepsilon = 0$), each oscillator is governed by

$$\ddot{\varphi} + V'(\varphi) = 0, \quad \Rightarrow \quad \frac{1}{2}\dot{\varphi}^2 + V(\varphi) = E,$$

where $\varphi \in H^2_{per}(0,T)$. The unique even solution $\varphi(t)$ satisfies the initial condition,

$$\varphi(0) = a, \quad \dot{\varphi}(0) = 0, \tag{6.5}$$

where a is the smallest positive root of $V(a) = E$ for a fixed value of E. Period of oscillations T is uniquely defined by the energy level E, according to the following formula:

$$T = \sqrt{2} \int_{-a}^{a} \frac{d\varphi}{\sqrt{E - V(\varphi)}}. \tag{6.6}$$

Theorem (Rothos & Pelinosky, '14)

Fix the period T *and the solution* $\varphi \in H^2_e(0,T)$ *of the nonlinear oscillator equation (??) with an even function* $V \in C^\infty(\mathbb{R})$ *such that* $V''(0) = 1$*. Assume that* $T \neq 2\pi n$, $n \in \mathbb{N}$ *and* $T'(E) \neq 0$*. Define* $\mathbf{Q}^{(0)}$ *by the representation (6.4) with fixed finite* $S \subset \mathbb{Z}$ *and* $\{\sigma_k\}_{k\in S}$*. There are* $\epsilon_0 \in \left(0, \frac{1}{2}\right)$ *and* $C > 0$ *such that for all* $\epsilon \in (-\epsilon_0, \epsilon_0)$*, there exists a unique solution* $\mathbf{Q}^{(\epsilon)} \in H^2_e((0,T), l^2(\mathbb{Z}))$ *of the discrete Klein–Gordon equation (5.2) satisfying*

$$\|\mathbf{Q}^{(\epsilon)} - \mathbf{Q}^{(0)}\|_{H^2_{\rm per}((0,T),l^2(\mathbb{Z}))} \leq C|\epsilon|. \tag{6.7}$$

Moreover, the map $(-\epsilon_0, \epsilon_0) \ni \epsilon \mapsto \mathbf{Q}^{(\epsilon)} \in H^2_e((0,T), l^2(\mathbb{Z}))$ *is* C^∞*.*

Floquet Multipliers

Linearize about the breather solution to the NMM by replacing $\mathbf{q}(t)$ with $\mathbf{Q}(t) + \mathbf{w}(t)$
Using the abstract evolution form (6.3) and the decomposition $M^{-1}(\epsilon) = I + \epsilon K(\epsilon)$, we can rewrite the linearized equations of KG-lattice in the equivalent form:

$$\frac{d^2\mathbf{w}}{dt^2} + \mathbf{f}'(\mathbf{Q})\mathbf{w} = -\epsilon K(\epsilon)\mathbf{f}'(\mathbf{Q})\mathbf{w}, \tag{7.1}$$

where $\mathbf{f}'(\mathbf{Q})$ is the diagonal operator with entries $V''(Q_n)$, $n \in \mathbb{Z}$.
$\mathbf{Q}(t+T) = \mathbf{Q}(t)$, an infinite-dimensional analogue of the Floquet theorem applies and the Floquet monodromy matrix $\mathcal{M}$ is defined by $\mathbf{w}(T) = \mathcal{M}\mathbf{w}(0)$.
The breather is stable if all eigenvalues of $\mathcal{M}$, called Floquet multipliers, are located on the unit circle and it is unstable if there is at least one Floquet multiplier outside the unit disk.

Floquet multipliers $\mu = e^{\lambda T}$ are found from solutions $\mathbf{W} \in H^2_{\rm per}((0,T), l^2(\mathbb{Z}))$ of

$$\frac{d^2\mathbf{W}}{dt^2} + 2\lambda\frac{d\mathbf{W}}{dt} + \lambda^2\mathbf{W} + \mathbf{f}'(\mathbf{Q})\mathbf{W} = -\epsilon K(\epsilon)\mathbf{f}'(\mathbf{Q})\mathbf{W}. \tag{7.2}$$

Floquet multiplier $\mu = 1$ corresponds to the characteristic exponent $\lambda = 0$. The generalized eigenvector $\mathbf{Z}_0 \in H^2_{\rm per}((0,T), l^2(\mathbb{Z}))$ of the eigenvalue problem (7.2) for $\lambda = 0$ solves the inhomogeneous problem,

$$\frac{d^2\mathbf{Z}_0}{dt^2} + \mathbf{f}'(\mathbf{Q})\mathbf{Z}_0 = -\epsilon K(\epsilon)\mathbf{f}'(\mathbf{Q})\mathbf{Z}_0 - 2\frac{d\mathbf{W}_0}{dt}, \tag{7.3}$$

where $\mathbf{W}_0$ is the eigenvector of (7.2) for $\lambda = 0$. To normalize $\mathbf{Z}_0$ uniquely, we add a constraint that $\mathbf{Z}_0$ is orthogonal to $\mathbf{W}_0$ with respect to the inner product

$$\langle \mathbf{W}_0, \mathbf{Z}_0 \rangle_{L^2_{\rm per}((0,T),l^2(\mathbb{Z}))} := \int_0^T \sum_{n\in\mathbb{Z}} (\bar{\mathbf{Z}}_0)_n(t)(\mathbf{W}_0)_n(t)dt.$$

Theorem (Pelinovsky & Rothos'14)

Under assumptions of Proposition 5, let $\mathbf{Q}^{(0)} = \sum_{k=1}^{N} \sigma_k \varphi \mathbf{e}_k$ *and* $\mathbf{Q}^{(\epsilon)} \in H_e^2((0,T), l^2(\mathbb{Z}))$ *be the corresponding solution of the discrete Klein–Gordon equation (5.2) for small* $\epsilon > 0$*. Then the eigenvalue problem (7.2) for small* $\epsilon > 0$ *has* $2N$ *small eigenvalues,*

$$\lambda = \epsilon^{1/2} \Lambda + \mathcal{O}(\epsilon),$$

where Λ *is an eigenvalue of the matrix eigenvalue problem*

$$\frac{T^2(E)}{T'(E) M_1} \Lambda^2 \mathbf{c} = \mathcal{S} \mathbf{c}, \quad \mathbf{c} \in \mathbb{C}^N. \tag{7.4}$$

Theorem cont.

Here the numerical coefficient M_1 is given by

$$M_1 = \int_0^T \ddot{\varphi}^2 dt > 0$$

and the matrix $\mathcal{S} \in \mathbb{M}^{N\times N}$ is given by

$$\mathcal{S} = \begin{bmatrix} -\sigma_1\sigma_2 & 1 & 0 & \dots & 0 & 0 \\ 1 & -\sigma_2(\sigma_1+\sigma_3) & 1 & \dots & 0 & 0 \\ 0 & 1 & -\sigma_3(\sigma_2+\sigma_4) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\sigma_{M-1}(\sigma_{M-2}+\sigma_M) & 1 \\ 0 & 0 & 0 & \dots & 0 & -\sigma_M\sigma_{M-1} \end{bmatrix}.$$

Travelling Waves in Nonlinear Magnetic Metamaterials Lattice

Lazarides *et al* PRL97 (2006) studied the discrete breathers in Nonlinear Megnetic Metamaterials, described by a dissipative lattice equation.
They consider a planar 1D array of N identical split-ring resonator (SSR) with their axes prependicular to the plane. The dynamics of charge q_n and the current i_n circulating in the nth SSR is described by

$$\begin{aligned} \frac{dq_n}{dt} &= i_n, \qquad n \in \mathbb{Z} \\ \frac{d}{dt}\left(\lambda i_{n-1} - i_n + \lambda i_{n+1}\right) &= \gamma i_n - f(t) + (q_n) \end{aligned} \tag{7.5}$$

with loss coefficient γ, coupling parameter λ, external forcing f and nonlinear function .
We investigate the existence of small periodic solutions of system (7.5) with periodic forcing, and the bifurcation of periodic solutions of (7.5) with small γ, λ and f.

Existence of small periodic solutions

Consider the equivalent equation

$$\lambda \ddot{q}_{n+1} - \ddot{q}_n + \lambda \ddot{q}_{n-1} = \gamma \dot{q}_n + (q_n) + f \cos(\omega t + pn) \tag{7.6}$$

where $\gamma \geq 0$, $\lambda \in \mathbb{R}$, $\omega > 0$, $f \neq 0$, $p \neq 0$ are parameters and is an odd analytic function with radius of convergence $\rho > 0$ such that $\mathrm{D}(0) = 0$.

$$q_n(t) = U(\omega t + pn), \quad U(z + \pi) = -U(z), \quad z = \omega t + pn$$

Existence of small periodic solutions

Consider the equivalent equation

$$\lambda \ddot{q}_{n+1} - \ddot{q}_n + \lambda \ddot{q}_{n-1} = \gamma \dot{q}_n + (q_n) + f \cos(\omega t + pn) \tag{7.6}$$

where $\gamma \geq 0$, $\lambda \in \mathbb{R}$, $\omega > 0$, $f \neq 0$, $p \neq 0$ are parameters and is an odd analytic function with radius of convergence $\rho > 0$ such that $\mathrm{D}(0) = 0$.

$$q_n(t) = U(\omega t + pn), \quad U(z + \pi) = -U(z), \quad z = \omega t + pn$$

$$\omega^2(\lambda U''(z+p) - U''(z) + \lambda U''(z-p)) = \gamma \omega U'(z) + (U(z)) + f \cos z. \tag{7.7}$$

We consider the associated Banach spaces and we rewrite the equation (7.7) as

$$\mathcal{K}U = \mathcal{F}(U, f)$$

$$\mathcal{K}U := \omega^2(\lambda U''(z+p) - U''(z) + \lambda U''(z-p)) - \gamma\omega U'(z), \quad \mathcal{F}(U, f) := (U) + f\cos z,$$

Theorem (VR *et al*, 2012)

Assume

$$\Theta := \inf_{k\in\mathbb{Z}} \sqrt{\omega^4(2k+1)^2\left(2\lambda\cos(2k+1)p - 1\right)^2 + \gamma^2\omega^2} > 0 \tag{7.8}$$

along with $|f| < |f_l|$ *for* f_l *satisfying*

$$A(r) := \Theta r - \sum_{k=3}^{\infty} \frac{|\mathrm{D}^k(0)|}{k!} r^k = |f_l|, \quad A'(r) = 0. \tag{7.9}$$

for some $r \in (0, \rho)$*. Then equation* (7.7) *has a unique solution* $U(f) \in \overline{B(\rho_f)}$ *in a closed ball where* $\rho_f < \rho$ *is the smallest positive root of* $A(r) = |f|$*. Moreover,* $U(f)$ *can be approximated by an iteration process.*

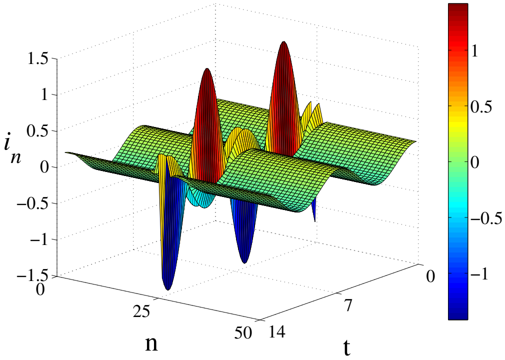
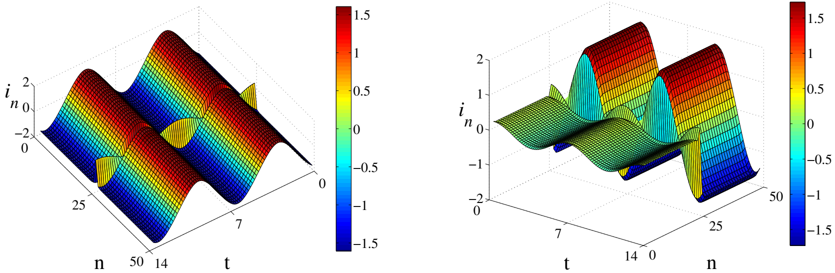


Figure: Time evolution of a one-site dissipative breather during approximatelly two periods for $T_b = 6.82, \lambda = 0.02, \gamma = 0.01, \varepsilon_0 = 0.04, \alpha = +1, \varepsilon_l = 2$ and $N = 50$. (Lazaridis *et al*, PRL 2006)

Dissipative Breathers



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Bifurcation Results

Consider (7.5) with small γ, λ and f. So we consider the system

$$\begin{aligned} \frac{dq_n}{dt} &= i_n, \quad n \in \mathbb{Z} \\ \frac{d}{dt}\left(\varepsilon\lambda i_{n-1} - i_n + \varepsilon\lambda i_{n+1}\right) &= \varepsilon\gamma i_n - \varepsilon h(\omega t + pn) + (q_n), \end{aligned} \tag{7.10}$$

for C^2-smooth and 2π-periodic $h, \in C^2(\mathbb{R}, \mathbb{R})$ and $\omega > 0$, $p \neq 0$, and $\varepsilon \neq 0$ is a small parameter.

Equation (7.10) implies

$$(\epsilon\lambda\ddot{q}_{n-1} - \ddot{q}_n + \epsilon\lambda\ddot{q}_{n+1}) = \epsilon\gamma\dot{q}_n - \epsilon h(\omega t + pn) + \varphi(q_n). \tag{7.11}$$

Putting $q_n(t) = U(\omega t + pn)$ for $U \in C^2(\mathbb{R}, \mathbb{R})$ in (7.11),

$$\omega^2 U''(z) + \varphi(U(z)) - \epsilon\lambda\omega^2\left(U''(z-p) + U''(z+p)\right) + \epsilon\gamma\omega U'(z) - \varepsilon h(z) = 0. \tag{7.12}$$

subharmonic Melnikov function

Theorem (VR *et al*, 2012)

Suppose $U''(z)+\varphi(U(z))=0$ *has a* $\overline{T}$*-periodic solution* U_0 *and* $T_\omega = 2\pi\frac{u}{v}$ *for* $u,v\in\mathbb{N}$. *If there is a simple zero* α_0 *of a Melnikov function*

$$M^{u/v}(\alpha) := \int_0^T \left(-\gamma\omega U'_\omega(z)+h(z+\alpha)\right) U'_\omega(z)dz, \tag{7.13}$$

then there is a >0 *such that for any* $0\neq\varepsilon\in(-\delta,\delta)$ *there is a unique* $2\pi u$*-periodic solution* $U(z)$ *of advance-delay DE with*

$$U(z)=U_0\left(\frac{z-\alpha_0}{\omega}\right)+O(\epsilon).$$

Theorem (VR *et al*, 2012)

Suppose $\varphi(0) = 0$, $\varphi'(0) < 0$ *and* $U''(z) + \varphi(U(z)) = 0$ *has an asymptotic solution* $\Gamma \in C^2(\mathbb{R}, \mathbb{R})$ *such that* $\lim_{|z|\to\infty} \Gamma(z) = 0$ *and* $\lim_{|z|\to\infty} \Gamma'(z) = 0$. *If there is a simple zero* β_0 *of the Melnikov function*

$$M(\beta) := \int_{-\infty}^{\infty} \left(-\gamma'(z) + h(\omega z + \beta)\right)'(z)dz. \tag{7.14}$$

Then there is a $\theta > 0$ *such that for any* $0 \neq \varepsilon \in (-\theta, \theta)$ *there is a unique bounded solution* $U(z)$ *of (7.12) on* $\mathbb{R}$ *with*

$$U(z) = \Gamma\left(\frac{z - \beta_0}{\omega}\right) + O(\epsilon).$$

Numerical Simulations

To illustrate the theoretical results obtained, we have solved the governing equation, the advance-delay equation using a pseudo-spectral method. We express the solution U in a Fourier series

$$U(z) = \sum_{j=1}^{J} \left[A_j \cos\left((j-1)\tilde{k}z \right) + B_j \sin\left(j\tilde{k}z \right) \right], \tag{7.15}$$

where $\tilde{k} = 2\pi/L$ and $-L/2 < z < L/2$. The Fourier coefficients A_j and B_j are then found by requiring the series to satisfy (7.7) at several collocation points. Hence, $2J$ collocation points are required, which are chosen with uniform grid points.
It is important to note that the physically relevant range for the coupling parameter λ is $|\lambda| < 1/2$.

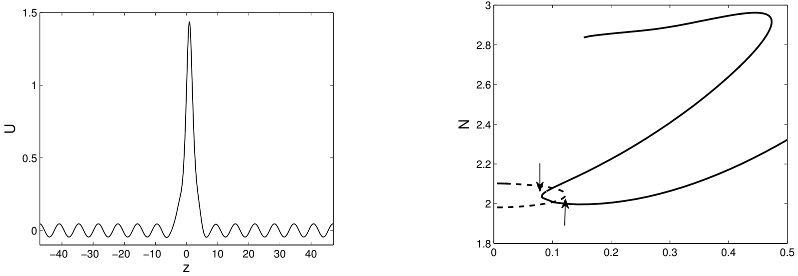

Figure: (1) An asymptotic travelling wave for a Duffing nonlinearity $\varphi = -U + U^3$, the profile of an asymptotic wave for $\lambda = \gamma = f = 0.1,\ \omega = 1$ and $p = \pi$. The single-hump profile is accompanied by periodic waves as suggested by Theorem. (2) Continuations of the solution in previous pic for varying γ (dashed) and f (solid). On the vertical axis is the solution norm $N = \sqrt{\int_{-L/2}^{L/2} |U|^2\, dx}$

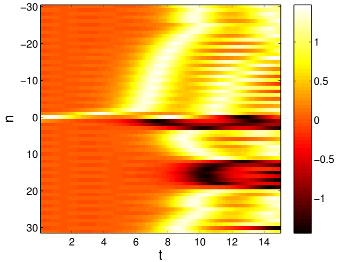

Figure: Time dynamics of the wave shown in 5. One can observe that the travelling wave is strongly unstable. The hump could only travel for one site before the background becomes excited and destroys the localised profile. The instability is naturally expected due to the fact that the zero solution (when there is no drive) forming the background of the asymptotic wave is unstable, i.e. it is a saddle point. For that reason, we believe that all the branches correspond to unstable solutions.

- D. Pelinosky and V Rothos Stability of discrete breathers in magnetic metamaterials in R. Carretero-Gonzalez et al. (eds.), Localized Excitations in Nonlinear Complex Systems,Nonlinear Systems and Complexity 7, Springer International Publishing Switzerland 2014.
- M. Feckan, M. Pospivsil, V. Rothos, H. Sussanto Travelling waves in nonlinear magnetic metamaterials in R. Carretero-Gonzalez et al. (eds.), Localized Excitations in Nonlinear Complex Systems,Nonlinear Systems and Complexity 7, Springer International Publishing Switzerland 2014.
- M. Feckan, M. Pospivsil, V. Rothos, H. Sussanto Periodic travelling waves of forced FPU lattices, J. Dyn. Diff 2013.
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- M. Agaoglou,V.M. Rothos, H. Susanto "Homoclinic chaos in a pair of parametrically-driven coupled SQUIDs", J. Phys.: Conf. Ser. 574 012027 doi:10.1088/1742-6596/574/1/012027.

1. **Periodic solutions in advanced-retarded differential equations**
   - Periodic Boundary Value Problem for functional differential equations.
   - Librational and Periodic travelling waves;
   - Multiplicity results.
2. **Travelling Waves in 2D Lattices: Mathematical Formulation;**
3. **Applications in travelling waves in nonlinear lattices**
4. **Travelling Waves in 1D Lattices: Mathematical Formulation;**
   - Moving Kinks for 1D lattice sine-Gordon
   - Numerical Simulations

Travelling Waves in 1D Lattice sine-Gordon

Frenkel Kontorova (FK) lattices have been studied as models for atomic chains, dislocations, charge density waves, magnetic and ferromagnetic domain walls in condensed matter physics and for parallel coupled one-dimensional Josephson junction arrays.
The potentials involved are chosen such that the continuum model supports both stationary and moving defects (kinks or anti-kinks) with topological charge $Q = 1$. That is, the kinks connect 0 to 2π (or vice versa) in the usual dimensionless form of potential adopted in the literature – the so-called sine-Gordon lattice. The discrete sine-Gordon

$$\ddot{u}_n(t) = u_{n+1}(t) - 2u_n(t) + u_{n+1}(t) - \Gamma^2 \sin u_n(t)$$

with solutions

- **Discrete kinks** (stationary solutions)
- **Moving discrete kinks** $u_n(t) = U(n - vt)$
- **Discrete Breathers** a highly spatially localized, time-periodic, stable (or at least very long-lived) excitation in a spatially extended.

Methodology

- The travelling wave equation of the corresponding dSG is formulated as a mixed-type differential equation.
- Applying dynamical system methods (center manifold, normal form) we focus on a 4D dynamical system,
- persistence of periodic solutions for the 4D system implies the existence of travelling waves with non-small amplitude oscillations on infinite nonlinear lattice,
- Analytical results are compared with numerical simulations for a concrete perturbed discrete nonlinear sine-Gordon equation, (*Rothos & Feckan 2005, Aigner, Champneys & Rothos, 2003*).

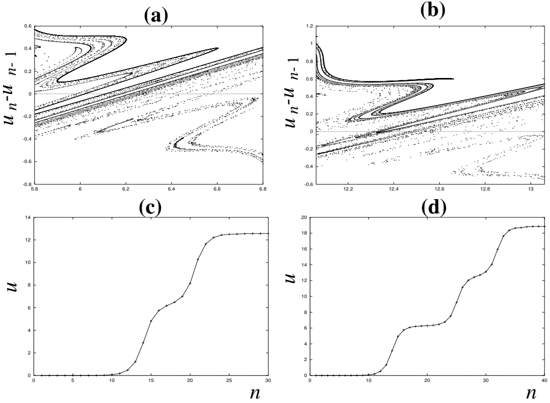


Figure: Construction of stationary kinks

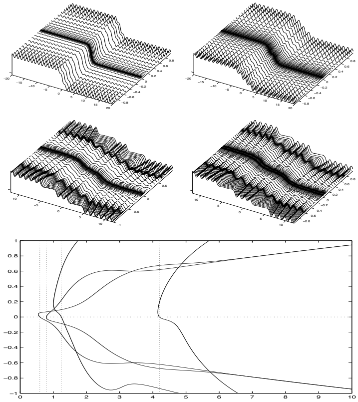
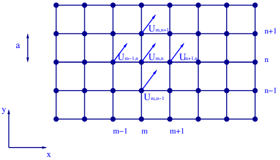


Figure: Travelling Kinks with tails

2D nonlinear lattices

An isotropic two dimensional planar model where rigid molecules rotate in the plane of a square lattice of spacing a.
At site (n, m) the angle of rotation is $u_{n,m}$ each molecule interacts linearly with its first nearest neighbors and with a nonlinear periodic substrate potential.
The equation of motion of the rotator at site (n, m) is

$$\ddot{u}_{n,m} = \mathrm{G}[u_{n+1,m} + u_{n-1,m} + u_{n,m+1} + u_{n,m-1} - 4u_{n,m}] + \omega_0^2 \sin u_{n,m}$$



where G the linear coupling coefficient and ω_0^2 square of the frequency of small oscillations in the bottom of the potential wells.

Due to the symmetry imposed by the lattice $\mathbb{Z}^2$, the existence and speed of a wave generally will depend on the direction $\mathrm{e}^{\mathrm{i}\theta}$ of motion.
Let $\theta \in \mathbb{R}$ be given, consider solution of lattice

$$u_{n,m}(t) = U(n\cos\theta + m\sin\theta - \nu t)$$

for some $\nu \in \mathbb{R}$ and $U : \mathbb{R} \to \mathbb{R}$.
Mixed-Type functional differential Equation $\nu \neq 0$:

$$\begin{aligned} \nu^2 U''(z) &= U(z+\cos\theta) + U(z-\cos\theta) + U(z+\sin\theta) + U(z-\sin\theta) \\ &\quad -4U(z) - f(U(z)) \end{aligned}$$

with $z = n\cos\theta + m\sin\theta - \nu t$ and $U(-\infty) = 0, \qquad U(+\infty) = 2\pi$

Theorem (Rothos & Feckan '07)

For any $\omega > 16$ *and* $1.17196 < T < 1.7579$, $2d$ *discrete sine-Gordon equation*

$$u_{n,m} - u_{n+1,m} - u_{n-1,m} - u_{n,m+1} - u_{n,m-1} + 4u_{n,m} + \omega \sin u_{n,m} = 0$$

possesses 4 *nontrivial/nonconstant travelling wave solutions of the form*

$$u_{n,m}(t) = \pi + U\Big(\frac{1}{\sqrt{2}}\Big(n+m\Big) - \frac{1}{2}t\Big)$$

or $U(z)$ *satisfying periodical conditions*

$$U(z+T) = U(z) + 2\pi, U(-z) = -U(z), T > 0, \text{ or}$$

$$U(z+T) = -U(z) + 2\pi, \text{ or } U(z+T) = -U(z), \text{ or, } U(z+T) = U(z), \ U(-z) = -U(z)$$

Theorem (Rothos & Feckan '07)

1. *Let* $\nu > 1$ *and* $f'(0) > 0$. *Moreover, suppose*

   e1) $\nu^2 \neq g_\theta\left(\frac{\pi}{T}k\right) + \frac{T^2}{4\pi^2 k^2} f'(0)\ \forall k \in \mathbb{N}$,

   e2) $\#\left\{k \in \mathbb{N} \mid \nu^2 < g_\theta\left(\frac{\pi}{T}k\right) + \frac{T^2}{4\pi^2 k^2} f'(0)\right\} \geq \left[\frac{T\sqrt{L}}{2\pi\sqrt{\nu^2 - 1}}\right] \geq 2$, *where* $[\cdot]$ *is the integer part function.*

   Then the advance-delay equation has at least 2 *nonzero odd* T*-periodic solutions.*

2. *Let* $\nu_1 < \nu < 1$. *Then the advance-delay equation for* $\theta = \pi/4$ *has infinitely many odd* π/r_ν*-periodic solutions* $\{U_n(z)\}_{n\in\mathbb{N}}$ *with*

$$|U_n(z) - c_n \sin 2r_\nu z| \leq \tilde{K}|\epsilon|$$

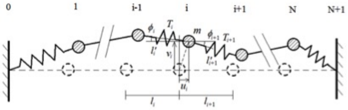
for $c_n \to \infty$ *as* $n \to \infty$ *and a constant* $\tilde{K} > 0$.

Nonlocal Nonlinear Lattice

A nonlinear lattice composed of a finite number of particles coupled by linear springs, executing in-plane oscillations, and having fixed boundary conditions, (Manevitch and Vakakis, 2014) demonstrates that the axial oscillations of the particles are an order of magnitude smaller than the transverse ones in the low-energy limit. This lattice as well as a granular chain without a pre-compression gives rise to a nonlinear sonic vacuum without any linear acoustic component, which is a medium with zero speed of sound as defined in classical acoustics (Nesterenko, 2001).

$$\begin{aligned} mU_i'' + (T_i - \xi\epsilon_i^{'})\cos\phi_i - (T_{i+1} - \xi\epsilon_{i+1}^{'})\cos\phi_{i+1} &= 0 \\ mV_i'' + (T_i - \xi\epsilon_i^{'})\sin\phi_i - (T_{i+1} - \xi\epsilon_{i+1}^{'})\sin\phi_{i+1} - F_i &= 0, \quad i = 1, K, N \end{aligned} \quad (9.1)$$

with U_i, V_i being the longitudinal and transversal displacements of i-th particle respectively, ϕ_i the angle between th spring and the horizontal direction, ξ the damping coefficeint, $\epsilon_i = l_i - l_i'$ the deformation of i-th spring, F_i the excited transverse force, and m the mass of each particle of the lattice.



Conclusions

- Overview of our recent theoretical and numerical activity in the theme of solitary nonlinear waves that arise in lattice system.
- The existence of localized traveling waves (sometimes called "moving discrete breathers" or "discrete solitons") in Klein-Gordon Lattices and discrete sine-Gordon
- We study the existence and bifurcation of discrete solitons in lattices with local and nonlocal interactions (Long range interactions).
- LS reduction method, perturbation method, dynamical systems method, topological and variational methods. Pseudospectral method, Numerical Bifurcation results.
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NLS: Introduction

The Nonlinear Schrödinger (NLS) equation

$$iu_t + u_{xx} + 2\sigma |u|^2 u = 0, \quad \text{where} \quad \sigma = \pm 1 \tag{10.1}$$

governs the dynamics of the envelopes of wavepackets in the dispersive media, and arises in many dierent contexts (nonlinear optics, water waves, etc.)
Zakharov and Shabat published the IST for the NLS equation. Then they extended the technique (ZS scheme) to some other equations (1973-1974). At about the same time, Ablowitz, Kaup, Newell and Segur (AKNS) developed an equivalent scheme, which generalizes the method, described earlier for the KdV equation (AKNS, 1974).

Lie symmetries of the NLS equation

In the following we use the following one-parameter groups of symmetries, admitted by the NLS equation (10.1):

- shift in t

$$t \to t + t_0, \quad x \to x, \quad u \to u$$

- shift in x

$$t \to t, \quad x \to x + x_0, \quad u \to u$$

- Galilean transformation

$$t \to t, \quad x \to x - ct, \quad u \to u \exp\left[i\frac{c}{2}\left(x - \frac{c}{2}t\right)\right]$$

- scaling

$$t \to a^2 t, \quad x \to ax, \quad u \to \frac{u}{a}$$

For example, if $u(x;t)$ is a solution of (10.1), then due to the Galilean invariance so are

$$u(x - ct, t)\exp\left[i\frac{c}{2}\left(x - \frac{c}{2}t\right)\right],$$

and so on.

Solitary waves of the NLS equation

We look for a solution of the NLS equation (10.1) of the form

$$u(x,t) = a(x)\mathrm{e}^{i\phi(t)}, \tag{10.2}$$

Substituting (10.2) into(10.1), we derive

$$-a\phi_t + a_{xx} + 2\sigma a^3 = 0. \tag{10.3}$$

Separating variables in (10.3) gives $\phi_t = \frac{a_{xx}}{a} + 2\sigma a^2 = const$. Then, integrating, we obtain (up to the scaling and the shift in t):

$$\phi = st$$

(we can assume that $s = \pm 1$), and

$$a_{xx} = -2\sigma a^3 + sa. \tag{10.4}$$

Multiplying (10.4) by a_x and integrating, we arrive at

$$(a_x)^2 = -\sigma a^4 + sa^2 + C.$$

It turns out that the form of solitary waves depends on the sign of (the sign of the nonlinear term in the NLS equation).

Focusing NLS: bright solitons

Case I ($\sigma = 1$, focusing NLS in the context of optics-"anomalous dispersion")

$$iu_t + u_{xx} + 2\left|u\right|^2 u = 0,$$

In this case, $(a_x)^2 = -a^4 + sa^2 + C$. If $a, a_x \to 0$ as $x \to \pm\infty$, then $C = 0$, and

$$\int \frac{da}{a\sqrt{s - a^2}} = \int dx.$$

For $s = 1$ we obtain the simplest form of the so-called bright soliton $a = \mathrm{sech} x, \phi = t$, yielding $u = \mathrm{e}^{it}\mathrm{sech} x$. (Consider the second case, $s = -1$.) Using the scaling and Galilean symmetries, we immediately obtain the two-parameter family of bright solitons:

$$u = A \mathrm{sech} A(x - ct) \exp\left[i\left(\frac{c}{2}x + A\left(A^2 - \frac{c^2}{4}\right)t\right)\right].$$

Note that A and c are independent parameters. (Two more parameters can be added using shifts in x and t, but these parameters are insignificant.)

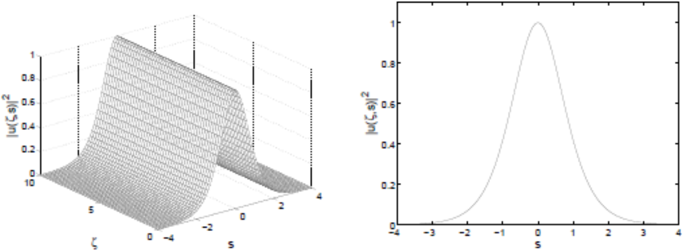

Figure 8. The fundamental bright soliton solution to the NLSE.

Defocusing NLS: dark solitons

Case II ($\sigma = -1$, defocusing NLS, in the context of optics-"normal dispersion")

$$iu_t + u_{xx} - 2|u|^2 u = 0,$$

In this case, $(a_x)^2 = a^4 + sa^2 + C.$, and solitary waves are rather different from those in Case I. For $s = -1$, choosing $C = 1/4$ (when the polynomial has repeated roots) we obtain the simplest form of the so-called dark soliton $a = \frac{1}{\sqrt{2}} \tanh \frac{x}{\sqrt{2}}, \phi = -t$, yielding

$$u = e^{-it} \frac{1}{\sqrt{2}} \tanh \frac{x}{\sqrt{2}}$$

(Consider the second case, $s = 1$.) Again, using symmetries, we obtain the two-parameter family of dark solitons:

$$u = \frac{A}{\sqrt{2}} \tanh \frac{A(x - ct)}{\sqrt{2}} \exp\left[i\left(\frac{c}{2}x - \left(A^2 + \frac{c^2}{4}\right)t\right)\right].$$

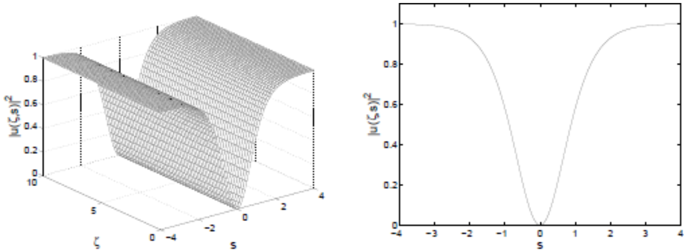

Figure 9. The fundamental dark (black) soliton solution to the NLSE.

Focusing NLS: Breathers

The focusing NLS equation models the evolution of one-dimensional packets of surface gravity waves on suciently deep water (Zakharov 1968). Recently, there has been renewed interest in the so-called "breather" solutions of this equation, which have been suggested as models for so-called "freak" waves (also, "rogue" waves). Loosely speaking, a "freak" wave is a single wave or a very short- and short-lived group with a significantly larger steepness than the surrounding waves.

NLS breather

The first breather type solution for the focusing NLS equation was found by Ma (1979). Ma solved the IVP for this equation, where the initial condition was a perturbed plane wave with boundary conditions $|q(x,t)| \to |q_0|$ as $x \to \pm\infty$. Ma has found that the asymptotic state for his problem consisted of a series of breathers (Ma-breathers), given below, and small dispersive radiation:

$$u_M = \frac{\cos(\Omega t - 2ik) - \cosh(k)\cosh(px)}{\cos(\Omega t) - \cosh(\phi)\cosh(px)} e^{2it}$$

Here, k is the real valued parameter, $\Omega = 2\sinh(2k)$ and $p = 2\sinh(k)$.
Taking the limit $k \to 0$ (i.e. when the breathing period tends to zero), Peregrine (1983) has obtained

$$u_P = \lim_{k\to 0} q_M = \left[1 - \frac{4(1+4it)}{1+4x^2+16t^2}\right] e^{2it}$$

Other breather-type solutions have been found by Akhmediev et al. (1987) and Ablowitz and Herbst (1990).

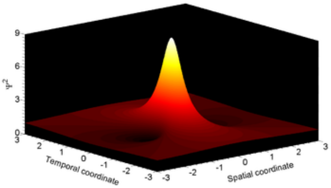

Figure: The Peregrine breather

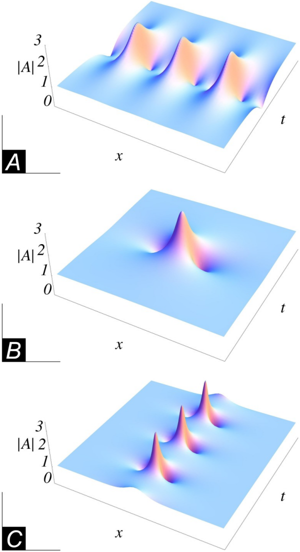

Figure: (A) The Akhmediev breather, (B) the Peregrine breather and (C) the KuznetsovâĂŞMa breather

Existence

We consider the NLS in abstract form:

$$iu_t + \Delta u + f(|u|^2)u = 0, \quad x \in \mathbb{R}^n, t \in \mathbb{R}$$

with prescribed initial data

$$u(x,0) = u_0(x)$$

Δ Laplace operator, smoothness of complex function $f(|u|^2)u : \mathbb{C} \to \mathbb{C}$. The subsititution of the general representation for standing waves $u(x,t) = e^{i\omega t}\phi(x), \omega \in \mathbb{R}$, into the NLS leads to the stationary equation

$$\Delta\phi - \omega\phi + f(\phi^2)\phi = 0, \quad x \in \mathbb{R}^n, \phi = \phi(x), \quad \phi(x)\,|_{|x|\to\infty} = 0.$$

We consider $f(\phi^2) = |\phi|^{p-1}, p > 1$.

Theorem

Let $\omega > 0, N > 3$ be integer, and $p \in \left(1, \frac{N+2}{N-2}\right)$. Then the stationary problem for NLS has a positive radial solution and, for any $l = 1, 2, 3, \ldots$, a radial solution $u_l = u_l(r)$, where $r = |x|$. with precisely l roots on the half-line $r > 0$. Let H be a real Hilbert space with a norm $\|\cdot\|$, J be a continuously differentiable real-value functional on H, and $S = \{h \in H : \|h\| = 1\}$. Let $r(\cdot) > 0$ be a continuously differentiable function on S such that for any $v \in S, J_r'(rv)|_{r=r(v)} = 0$. Then, if the functional $\hat{J}(v) = J(r(v)v)$ considered on S has a critical point $v_0 \in S$, then $J'(h)|_{h=r(v_0)v_0} = 0$.

$$J(\phi) = \int_{\mathbb{R}^n} \left\{ \frac{1}{2} \left(|\nabla \phi|^2 + \omega \phi^2 \right) - \frac{1}{p+1} |\phi|^{p+1} \right\} dx.$$

Concentration-Compactness Method (P.L.Lions)

Theorem

We consider the functional

$$E(u) = \int_{\mathbb{R}^n} \left\{ \frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} |u|^{p+1} \right\} dx.$$

under the restriction $|u|_2^2 = \lambda$, $\lambda > 0$ *is fixed, we set*

$$I_\lambda = Inf \left\{ E(u) \,/\, u \in H^1, |u|_2^2 = \lambda \right\}$$

Let $p \in \left(1, 1 + \frac{4}{N}\right)$ *and* $\lambda > 0$ *be arbitrary. Then* $I_\lambda > -\infty$ *and for an arbitrary minimizing sequence* $\{u_n\}_{n=1,2,3,\ldots}$ *of the minimizing problem there exists a sequence* $\{y_n\}_{n=1,2,3,\ldots} \subset \mathbb{R}^n$*, such that the sequence* $\{u_n(\cdot + y_n)\}_{n=1,2,3\ldots}$ *is relatively compact in* H^1 *and its arbitrary limit point is a solution of minimization problem.*

We have applied the above results to the problem of saturable Discrete NLS equation (Pankov & Rothos, 2008).

Stability

Theorem

Let $f(|u|^2)u$ be a continuously differentiable function of the complex argument u for NLS with $(N = 1)$. There exists a soliton-like solution $\bar{u}(x,t) = e^{i\omega t}\phi(\omega_0, x)$ for the NLS vanishing as $x \to \pm\infty$, for which there exists $\frac{\partial}{\partial\omega}\phi(\omega,\cdot)\Big|_{\omega=\omega_0} \in L_2$ and

$$\frac{d}{d\omega}P(\phi(\omega,\cdot))\Big|_{\omega=\omega_0} = 2\int_{-\infty}^{\infty}\phi(\omega_0,x)\phi'_\omega(\omega_0,x)dx.$$

If the condition

$$\frac{d}{d\omega}P(\phi(\omega,\cdot))\Big|_{\omega=\omega_0} > 0$$

is satisfied, then this soliton-like solution $\bar{u}(x,t)$ is stable.

Problem set-up

A model of nonlocal nonlinear Kerr-type media, in which the refractive index change $\Delta n(I)$ induced by a beam with intensity $I(x,t)$, can be represented in general form as

$$\Delta n(I) = \pm \int_{-\infty}^{+\infty} R(x'-x) I(x',t) \mathrm{d}x \tag{12.1}$$

$\pm$ sign corresponds to a focusing (defocusing) nonlinearity/ The real, localized, and symmetric function $R(x)$ is the response function of the nonlocal medium, whose width determines the degree of nonlocality. Here, the intensity $I(x,t)$ will be equal to $I(x,t) = |\Psi(x,t)|^2$ so we will have

$$i\partial_t \Psi + \frac{1}{2}\partial_x^2 \Psi + V(x)\Psi + \sigma \Delta n(I)\Psi = 0, \quad V(x) = V_0 \sin^2\left(\frac{\pi x}{d}\right), \tag{12.2}$$

When the nonlocality is weak, we can expand $I(x,t)$ around the point $x' = x$ to obtain

$$\Delta n(I) = \sigma(I + \gamma \partial_x^2 I) \tag{12.3}$$

where the nonlocality parameter $\gamma > 0$ is given by

$$\gamma = \frac{1}{2} \int_{-\infty}^{+\infty} R(x) x^2 \mathrm{d}x, \tag{12.4}$$

where response function $R(x)$ considering to be symmetric with $\int_{-\infty}^{+\infty} R(x)\mathrm{d}x = 1$. Substituting Eq. (12.3) into Eq. (12.2) yields the modified nonlinear Schrödinger equation

$$i\partial_t \Psi + \frac{1}{2}\partial_x^2 \Psi + V(x)\Psi + \sigma\left(|\Psi|^2 + \gamma \partial_x^2 |\Psi|^2\right)\Psi = 0, \tag{12.5}$$

Solitary waves of Eq. (12.5) are sought in the form

$$\Psi(x,t) = \Phi_s(x) e^{i\mu_s t}, \tag{12.6}$$

where μ is the propagation constant and $\psi(x)$ is an amplitude function that is localized in space.
Substituting the above equation into Eq. (12.5), we get

$$\frac{1}{2}\partial_x^2\Phi_s + V(x)\Phi_s - \mu_s\Phi_s + \sigma\left(|\Phi_s|^2 + \gamma\partial_x^2|\Phi_s|^2\right)\Phi_s = 0, \tag{12.7}$$

we need to use the hypothesis that the Bloch band around a band edge $\mu = \mu_0$ is real. Furthermore, in Figure 11 we show the effect of variable γ on the profile of an on-site gap soliton. In particular, as the value of $\gamma \to 0$ the amplitude of the on-site profile decreases. On the other hand, by increasing the value of γ we obtain an on-site soliton with higher amplitude.

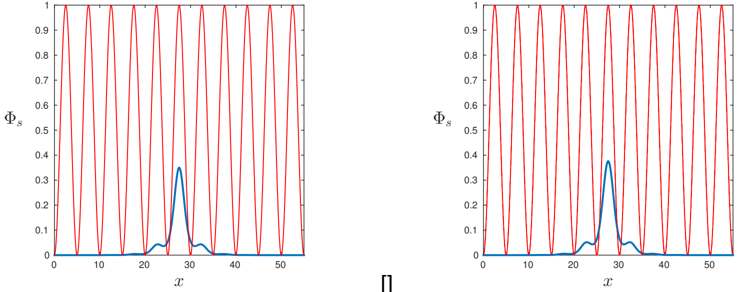

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Figure: Comparison of on-site gap soliton profiles for different values of variable γ. The red solid line represents function $V(x) = \sin(\frac{\pi x}{5})^2$. **(a)** On-site localised soliton for $\gamma = 0.01$ **(b)** By increasing the value of $\gamma = 0.3$ the amplitude of the on-site gap soliton increases.

The linear version of Eq. (12.7) is a Mathieu-type equation and admits two linearly independent solutions in the form

$$\psi_\eta(x,\mu) = e^{i\kappa x}\tilde{\psi}_\eta(x,\mu), \tag{12.8}$$

where $\tilde{\psi}_\eta(x,\mu)$ is periodic with the same period π as the potential $V(x)$ and κ lies in the first Brillouin zone $-1 \le \kappa \le 1$. The solution of Eq. (12.7) is expanded in powers of an amplitude parameter, $0 < \epsilon \ll 1$

$$\Phi_s = \epsilon A_\eta(X)\psi_\eta(x) + \epsilon^2 A'_\eta \psi_\eta^{(1)}(x) + \epsilon^3 \phi_s^{(2)}(x;X) + O(\epsilon^4), \tag{12.9}$$

$$\Phi_\epsilon = \frac{\Phi_s}{\epsilon} \qquad \mu_s = \mu_\eta + \epsilon^2\Delta_\eta + O(\epsilon^4), \tag{12.10}$$

where $X = \epsilon x$ is the 'slow' variable of the function $A_\eta(X)$. Substituting the expansion of Eq. (12.9) into Eq. (12.7) up to order $O(\epsilon^2)$ yields

$$\frac{1}{2}\psi_\eta^{(1)''} + (V(x) - \mu_s)\psi_\eta^{(1)} = -\psi'_\eta(x), \tag{12.11}$$

whence $\psi_\eta^{(1)}$ gives

$$\int_{-\pi}^{\pi} \psi'_\eta \psi_\eta^{(1)*} \mathrm{d}x = 0. \tag{12.12}$$

$A_\eta = A_\eta(X)$ satisfies the nonlinear equation:

$$A_\eta'' \mu_\eta^{(2)} - A_\eta \Delta_\eta + A_\eta^3 \chi_\eta^{(2)} = 0, \tag{12.13}$$

where

$$sgn(\mu_\eta^{(2)}) = sgn(\Delta_\eta) = sgn(\chi_\eta^{(2)}), \qquad \mu_\eta^{(2)} = \beta_\eta^2 \Delta_\eta \qquad \mu_\eta^{(2)} = \frac{\chi_\eta^{(2)} \alpha_\eta \beta_\eta^2}{2}, \tag{12.14}$$

and

$$\chi_\eta^{(2)} = \sigma \frac{\displaystyle\int_{-\pi}^{\pi} |\psi_\eta|^2 (|\psi_\eta|^2 + \gamma \psi_\eta'' \psi_\eta^* + 2\gamma \psi_\eta' \psi_\eta^{*\prime} + \gamma \psi_\eta \psi_\eta^{*\prime\prime}) \mathrm{d}x}{\displaystyle\int_{-\pi}^{\pi} |\psi_\eta|^2 \mathrm{d}x}, \tag{12.15}$$

Eq. (12.13) admits a sech-type solitary wave solution

$$A_\eta = \alpha_\eta \mathrm{sech}\left(\frac{X - X_0}{\beta_\eta}\right). \tag{12.16}$$

Melnikov Function

We proceed to compute the Melnikov function, according to which we first take the derivative of Eq. (12.7) in x, so we have the third-order ordinary derivative equation:

$$\frac{1}{2}\Phi_s''' + V'(x)\Phi_s + V(x)\Phi_s' - \mu_s\Phi_s' + \sigma\Phi_s'\Phi_s^2 + \sigma\Phi_s^2\partial_x\Phi_s^2 + \gamma\sigma\Phi_s'\partial_x^2\Phi_s^2 + \gamma\sigma\Phi_s\partial_x^3\Phi_s^2 = 0. \tag{12.17}$$

Multiplying Eq. (12.17) by Φ_s and integrating from $-\infty$ to $+\infty$ using integration by parts, we obtain the following constraint

$$M_s(x_0) = \int_{-\infty}^{+\infty} V'(x)\Phi_s^2 \mathrm{d}x = 0 \tag{12.18}$$

Substituting the perturbation expansion of Eq. (12.9) into the above constraint we get

$$M_s = \epsilon^2 \int_{-\infty}^{+\infty} A^2(X)V'(x)\psi_\eta^2(x)\mathrm{d}x + +2\epsilon^3 \int_{-\infty}^{+\infty} A(X)A'(X)V'(x)\psi_\eta(x)\psi_\eta^{(1)}(x)\mathrm{d}x + \ldots = 0. \tag{12.19}$$

We then calculate the integrals, starting with the first term. We expand the product $V'(x)\psi(x)^2$ into the following Fourier series expansion

$$V'(x)\psi_\eta^2(x) = \sum_{m=1}^{\infty} c_m \sin\left(\frac{2m\pi x}{d}\right), \tag{12.20}$$

where $c_1, c_2, \ldots$ are the Fourier coefficients.
Now we put Eq. (12.20) into Eq. (12.19), so that we may derive exponentially small terms to this integral, keeping only the leading order term for the first Fourier mode($m = 1$) and using Eq. (12.16), we find the integral $W_1 \sin(2x_0)$ where

$$W_1 = \epsilon c_1 \alpha_\eta^2 \int_{-\infty}^{+\infty} \operatorname{sech}^2\left(\beta_\eta X\right) \cos\left(\frac{2\pi X}{\epsilon d}\right) \mathrm{d}X = \frac{2\pi^2 c_1 \alpha_\eta^2}{\beta_\eta^2 d} \operatorname{csch}\left(\frac{\pi^2}{\epsilon \beta_\eta d}\right), \tag{12.21}$$

We now consider that $W_1 = O(e^{-\frac{\pi^2}{\epsilon\beta\eta d}})$, which is exponential small in ϵ, while all the terms in Eq. (12.19) are of the same order in ϵ. Hence we can easily prove using integration by parts that all coefficients W_η are of the same order in ϵ, and thus arrive at

$$M_s = \left(\sum_{n=1}^{\infty} W_n \right) \sin\left(\frac{2\pi x_0}{d}\right) = 0. \tag{12.22}$$

Notice here that the weak nonlocality influences the above equation through the parameter α_η(see Eq. (12.14)), which means that it will have an important effect on stability. Also the above constraint would be satisfied if

$$\sin\left(\frac{2\pi x_0}{d}\right) = 0 \quad \Rightarrow \quad x_0 = 0, \frac{d}{2} \tag{12.23}$$

In the first case ($x_0 = 0$) the corresponding soliton is called an on-site soliton because the peak of the envelope function A_η is located at a potential minimum whereas in the second case ($x_0 = d/2$) the corresponding soliton is called off-site soliton since the peak of the envelope function A_η is located at a potential maximum.

Symmetry Breaking Instability of Gap solitons

In order to study the linear stability of gap solitons Φ_s, near the band edge $\mu_s = \mu_n$, employing the method of linear stability analysis, we assume

$$\psi(x,t) = e^{-i\mu_s t}\Big[\Phi_s(x) + (u(x) + iw(x))e^{\lambda t} + (u^*(x) - iw^*(x))e^{\lambda^* t}\Big], \tag{12.24}$$

where $\epsilon \ll 1$, while u, w are the perturbation eigenfunctions and λ is the growth rate of the perturbation. Linearizing Eq. (12.7) we obtain

$$i\mathcal{L}\mathrm{W} = \lambda \mathrm{W}, \tag{12.25}$$

where $\mathrm{W} = [u, w]^T$

$$\mathcal{L} = \begin{pmatrix} 0 & \mathcal{L}_0 \\ -\mathcal{L}_1 & 0 \end{pmatrix}, \tag{12.26}$$

Here $\mathcal{L}_0$ and $\mathcal{L}_1$ are Schrödinger operators

$$\mathcal{L}_0 w = -\frac{1}{2}\partial_x^2 w - (V(x) - \mu_s)w - \sigma\Phi_s^2 w - \gamma\sigma(\partial_x^2\Phi_s^2)w,$$

$$\mathcal{L}_1 u = -\frac{1}{2}\partial_x^2 u - (V(x) - \mu_s)u - 3\sigma\Phi_s^2 u - \gamma\sigma(\partial_x^2\Phi_s^2)u - 2\gamma\sigma\Phi_s\partial_x^2(\Phi_s u). \tag{12.27}$$

The gap soliton Φ_s is therefore linearly unstable if λ has an imaginary component, while if λ is real the gap solitons is stable.

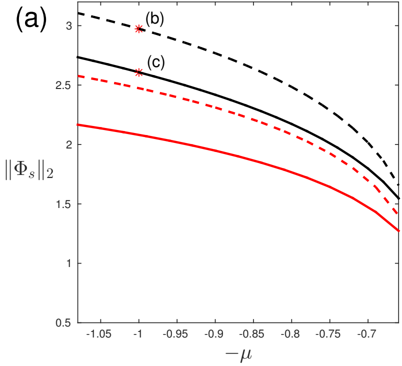


Figure: Bifurcation of the norm for off-site (dashed line) and on-site (solid line) solitons with respect to variable μ for two different γ values, $\gamma = 0.9$ (black lines) and $\gamma = 0.001$ (red lines). **(a)** Bifurcation diagram for both solitons. The points marked on the diagram correspond to a profile for each soliton.

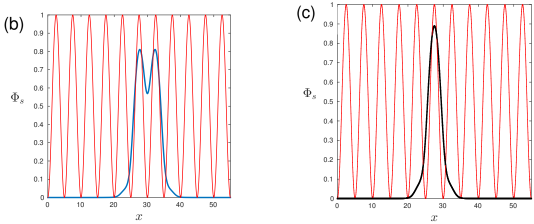


Figure: Bifurcation of the norm for off-site (dashed line) and on-site (solid line) solitons with respect to variable μ for two different γ values, $\gamma = 0.9$ (black lines) and $\gamma = 0.001$ (red lines). In both calculations we use the function $V(x) = \sin(\frac{\pi x}{5})^2$ which can be seen in (b) and (c) plotted in the red solid line and parameters $\sigma = 1$, $\gamma = 0.3$. **(b)** Profile of the off-site soliton solution. **(c)** Profile of the on-site soliton.

the symmetry of the modified NLS Eq. (12.7), we have a nonempty kernel of operators $\mathcal{L}_0$ and $\mathcal{L}_1$ at all power orders of ϵ^n, so

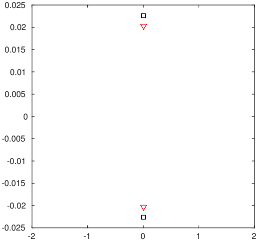
$$\mathcal{L}_0\Phi_\epsilon(x,X)=0, \qquad \mathcal{L}_1 U_\epsilon = 0(\epsilon^n), \qquad U_\epsilon = \frac{\partial\Phi_\epsilon(x,X)}{\partial X}, \tag{12.28}$$

Using straightforward calculations we can show below that the zero eigenvalue of $\mathcal{L}_1$, connected with the eigenfunction U_ϵ, shifts according to

$$\left\langle U_\epsilon, \mathcal{L}_1 U_\epsilon \right\rangle = \frac{1}{2\epsilon^4} M_s'(x_0), \tag{12.29}$$

If $\lambda = \lambda_\epsilon$ is a small eigenvalue corresponding for eigenfunction u_ϵ, we obtain that

$$\lambda_\epsilon^2 \approx -\frac{2M_s'(x_0)}{\epsilon^2 \left\langle \Phi_\epsilon, \Phi_\epsilon \right\rangle}. \tag{12.30}$$



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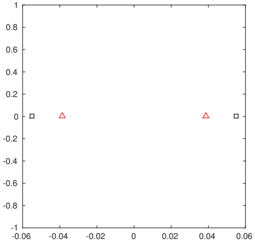

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Figure: Discrete eigenvalues for on-site and off-site soliton solutions for two different values of γ, $\gamma = 0.9$ (black squares) and $\gamma = 0.001$ (red triangles). **(a)** Two neutrally stable imaginary eigenvalues computed for the solution as seen in Fig.2 (b). **(b)** Two real unstable eigenvalues numerically computed for the solution as seen in Fig.2 (c).

Conclusions

- We briefly reviewed the solitonic solutions of NLS.
- We reviewed the theorems of existence and stability of solutions for NLS.
- NLS with weak nonlocality
    - We studied the stability of one-dimensional gap solitons employing the modified NLS equation with a sinusoidal potential together with the presence of a weak nonlocality.
    - Using Melnikov function, it is proved that two soliton families bifurcate out from every Bloch-band edge under self-focusing or self-defocusing nonlinearity, and one of these is always unstable.

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