Normal and Anomalous Diffusion (Tutorial)

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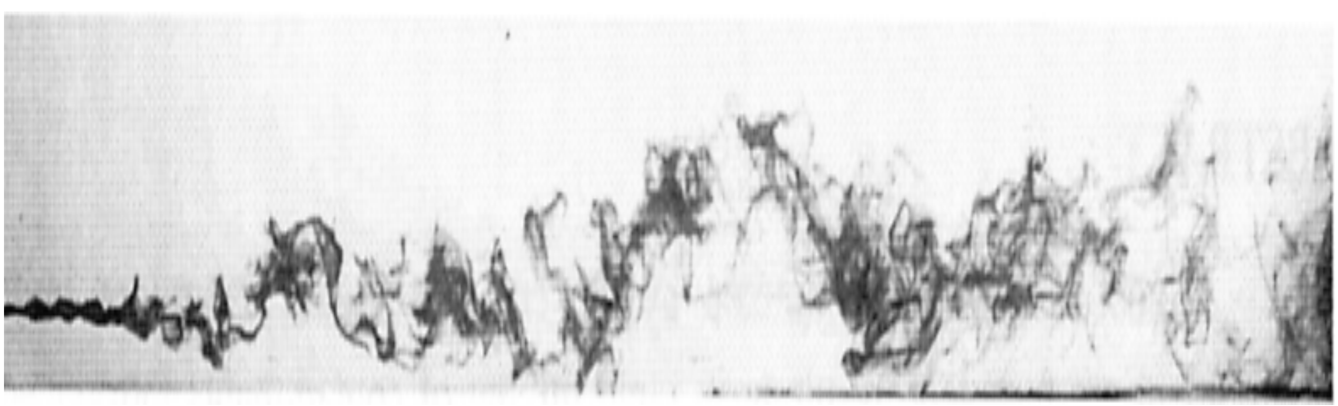
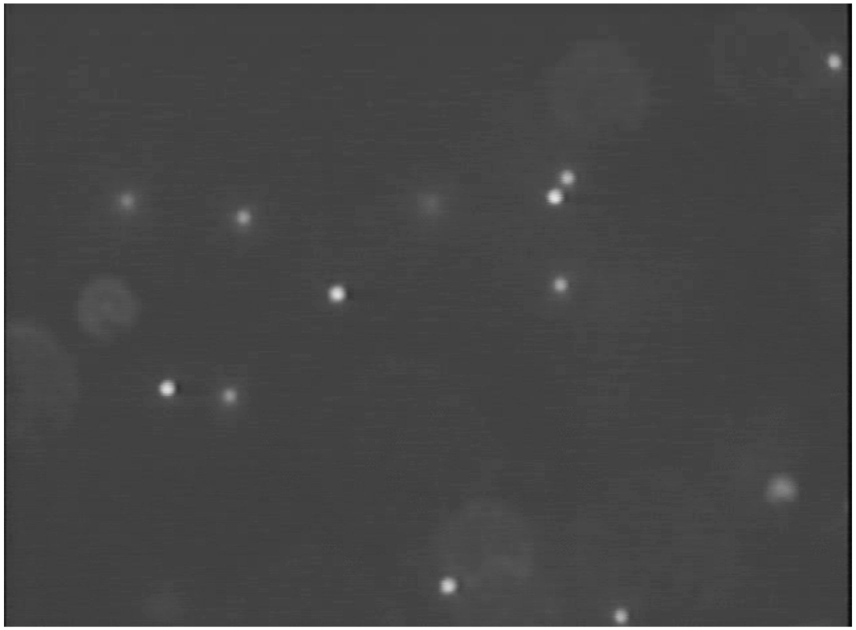


Topics

- Motivation
- Brownian motion and random Walks
- Normal Diffusion
- Walks on Fractal media-traps-Levy flights
- Anomalous diffusion
- Applications and open problems

Motivation



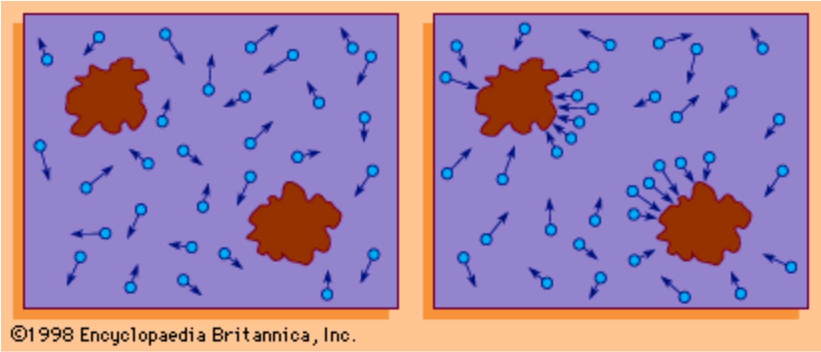
The art of doing research in physics

- We usually start with an observation of natural phenomenon
- We the have a nice idea on “How this phenomenon can be interpreted”
- We need model equations or simulation to build a solid base on the idea.
- Then the idea, started from an observation and moved on to a generic mathematical model, can become a prototype for interpreting many natural phenomena…. and this in the beauty of the scientific method ……

Back on the “Brownian motion” : the idea

- Observed first by Jan Ingenhousz 1785, but was rediscovered by Brown in 1828.
- Pollen grains (from trees, plans) are organic substances with life in them, the erratic motion is expression of the power inherent to life (the botanologist)-Brown
- Motion of small particles suspended in a fluid due to bombardment by molecules in thermal motion (the physicist)-Einstein.

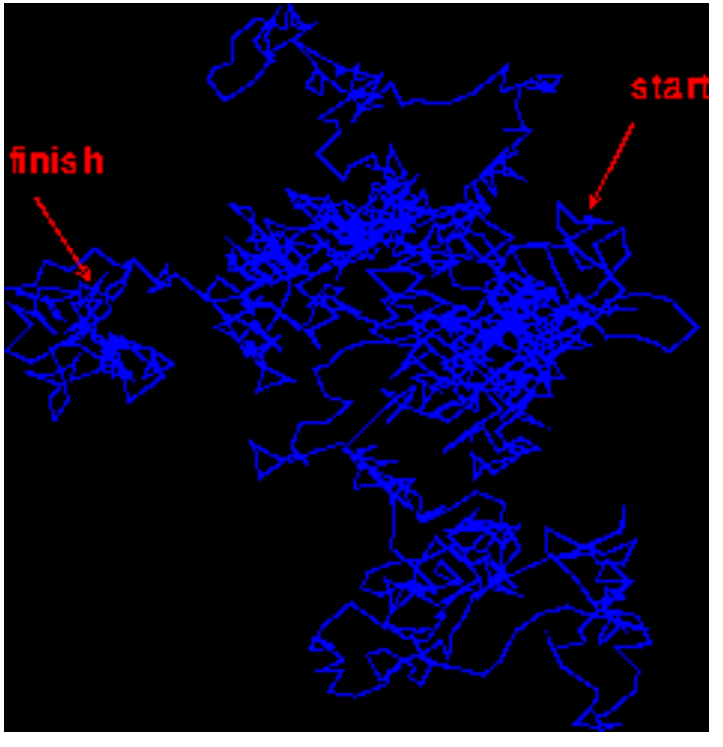
Qualitative Idea



Can we pose another question: How long it will take a drank man to go from the bar to his house?

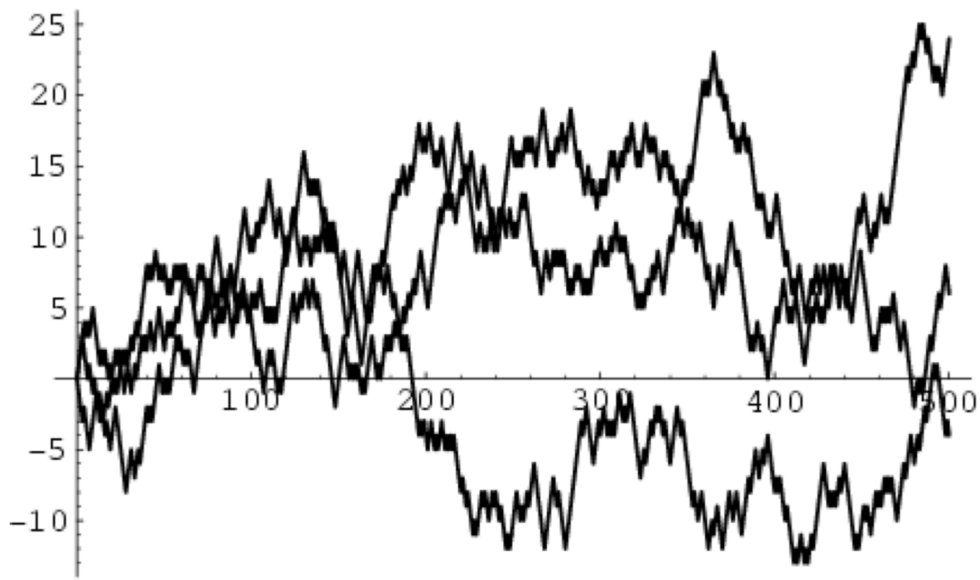
Random walk in 2D

- Choose a random value Δx in the interval [-1,1] and $\Delta y = \pm\sqrt{1-\Delta x^2}$



Question

- What will be the statistics of the distance <r(t_0)> at time t_0 after many repetitions?



More….

$$R^2 = (\Delta x_1 + \Delta y_1 + \Delta x_2 + \Delta y_2 + \Delta x_3 + \Delta y_3 + ... + \Delta x_N + \Delta y_N)^2$$

$$= (\Delta x_1)^2 + ...(\Delta y_N)^2 + ... + 2(\Delta x_1 \Delta x_2) + ...$$

$$< R >= \sqrt{N < r^2 >} = \sqrt{N} r_{rms}$$

- If the distance of the drank man from the bar to his house is 1000m and his step is 1m then you estimate the number of steps that are necessary and assuming that it takes several seconds for each step… you can estimate how long it will take him to reach home….

Mean free path

- A typical particle moving inside a fluid with density n of molecules with radius α will travel a mean distance

$$\lambda = < v > \tau$$

between collisions, <v> is the mean velocity and τ the collision time.

- Let us assume an ideal tube of length L and particle collision cross section α inside the fluid. Typical particle will suffer

$$N = 4\pi\alpha^2 Ln$$

Collisions before exiting. From this relation we estimate the mean free path

$$\lambda = \frac{1}{4\pi\alpha^2 n}$$

Diffusion from random collisions

$$< R^2 >= N(< r^2 >) = (t / \tau)(< r^2 >) = Dt$$

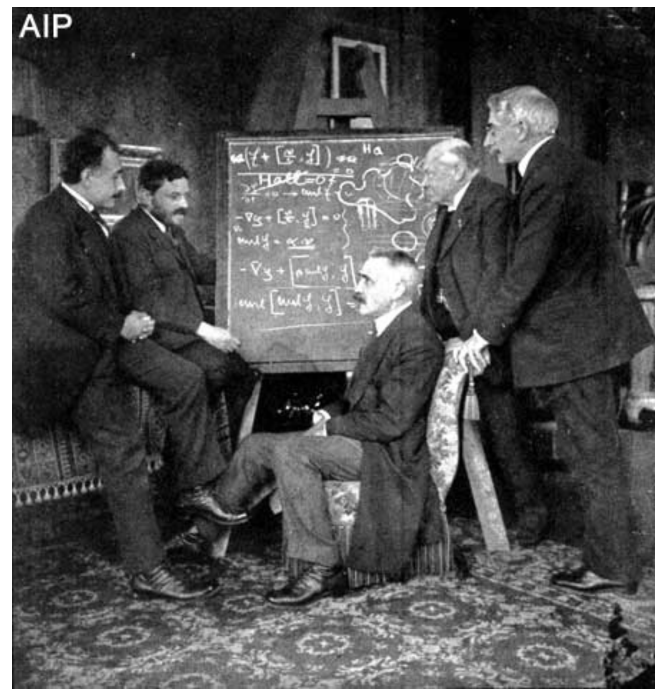
$$D \sim< r^2 > / \tau, \quad \tau = \lambda / v_{rms}$$

Mathematical formula for Brownian motion
Langevin Formula

- Paul Langevin at 1908 modeled the Brownian motion
- $m\,\ddot{x} = -\gamma\dot{x} \quad + R(t)$

m is the mass of the particle,
$\dot{x}$ its Speed, γ=6πηα,
η=dynamic viscosity,

R(t)=randomly Fluctuating force



More on Langevin's formula

$$m x\ddot{x} = m\left[\frac{d(x\dot{x})}{dt} - \dot{x}^2\right] = -\gamma x\dot{x} + xF(t)$$

$$m\left[\frac{d < x\dot{x} >}{dt} - < \dot{x}^2 >\right] = -\gamma < x\dot{x} > + < xF(t) >$$

$$< xF(t) > = < x >< F(t) > = 0$$

$$\frac{1}{2} m < \dot{x}^2 > = \frac{1}{2} kT$$

$$\left[\frac{d < x\dot{x} >}{dt} + \frac{\gamma}{m} < x\dot{x} >\right] = kT / m$$

$$< x\dot{x} > = \frac{1}{2}\frac{d}{dt} < x^2 > = \frac{kT}{\gamma}(e^{-\gamma t/m} + 1)$$

More on Langevin's formula

$$< x^2 >= \frac{2kT}{m}\left[t - \frac{m}{\gamma}(1 - e^{-\gamma t/m}) \right]$$

More on Langevin's formula

- For

$$t << (\gamma / m)^{-1}$$

$$< x^2 >= \frac{kT}{2m} \gamma t^2$$

- Ballistic

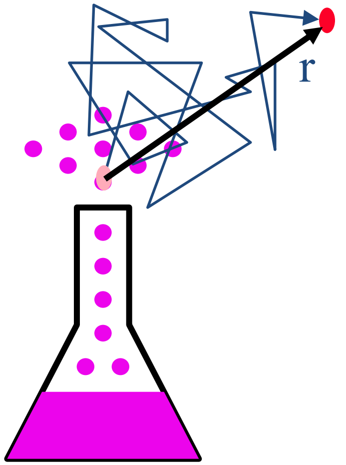
$$t >> (\gamma / m)^{-1}$$

$$\textit{Normal Diffusion}$$

$$< x^2 >= \frac{2kT}{\gamma} t = \frac{kT}{3\pi\eta\alpha} t$$

$$< r^2 >= 3 < x^2 >= \frac{kT}{\pi\eta\alpha} t = Dt$$

Exercise 1: Perfume



1. ***If the diffusion constant in atmosphere at 300 K isD = 10^{-5} m^2/s, how far (in <u>any</u> direction) will perfume particles diffuse in 1 minute?***

2. ***Approximately how far <u>up</u> will the perfume diffuse in 1 minute?***

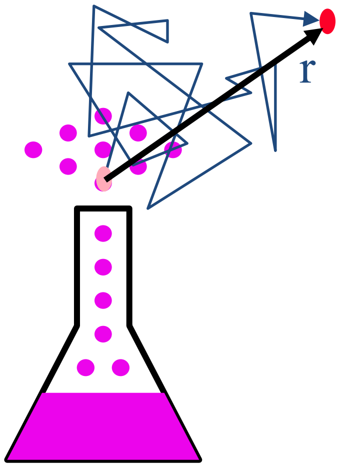
Exercise 1: Perfume

1.

$$r_{rms} \approx \left\langle r^2 \right\rangle^{1/2} = \sqrt{Dt}$$

$$= \sqrt{\left(10^{-5}\,m^2/s\right)\left(60s\right)}$$

$$\approx 6 \times 10^{-2}\,m = 6\,cm$$



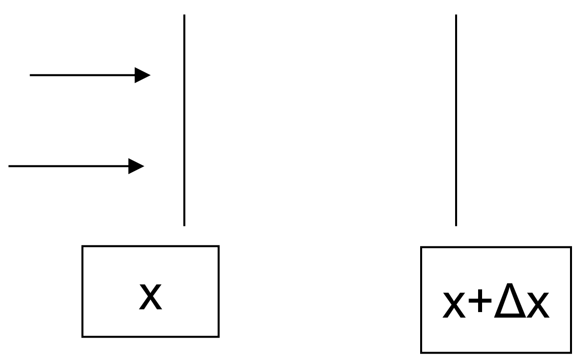
The Diffusion equation

- Fick's law
- The flux is proportional to the gradient in concentration

$$J = -D\frac{\partial n}{\partial x}$$

$$\frac{\partial n}{\partial t} = -\frac{\partial J}{\partial x}$$

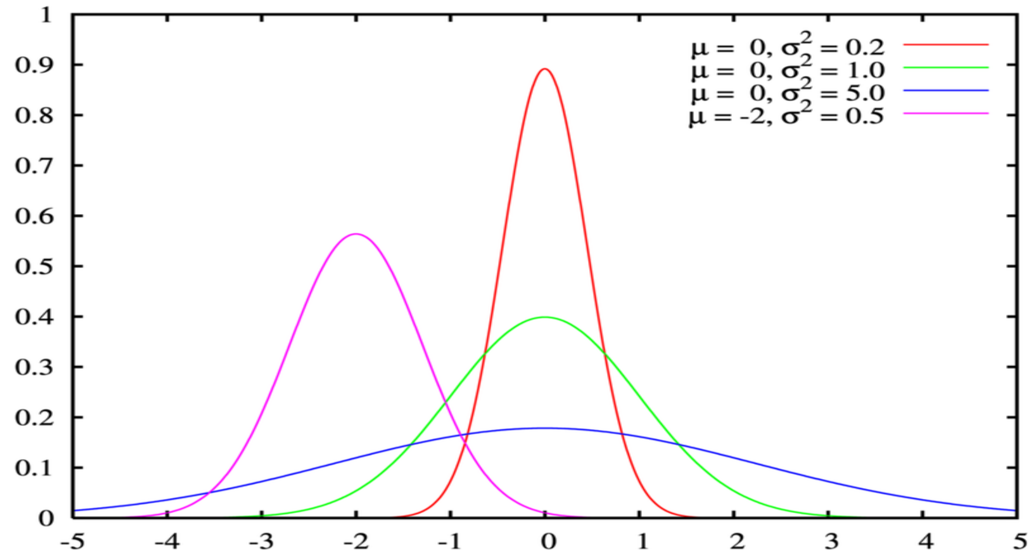
$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial t^2}$$



Solution of Diffusion Equation

$$n(x,0) = \delta(x)$$

$$n(x,t) = \frac{1}{\sqrt{4\pi D t}} e^{-\frac{x^2}{4Dt}}$$



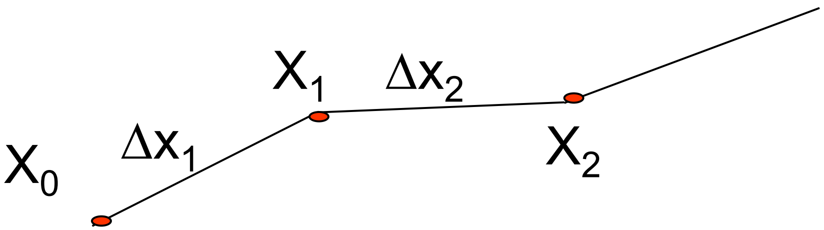
How to treat formally the classical RW

- Only position *x* of a particle is considered
- Time step Δt constant (time plays dummy role, a simple counter)
- Position of particle after n-steps (at time $t_n = n\Delta t$): x_n

$$x_n = \Delta x_n + \Delta x_{n-1} + \Delta x_{n-2} + \ \dots \ + \Delta x_1 + x_0$$

Δx_i: jump increment: random
x_0: initial position

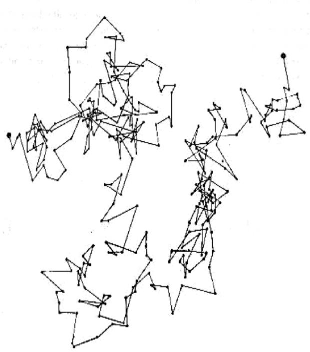
- Need to specify



- distribution of jump increments $q(\Delta x)$: prob. to make a jump Δx
- → RW completely specified:
problem: determine solution, i.e. probability $P(x,t_n)$ that a particle is at position *x* at time $t_n = n\ \Delta t$

How to treat formally the classical RW

- 1827: Brown observed that small particles (pollen grains) in a fluid followed an erratic zig-zag path when seen under the microscope:now called Brownian motion – prototype of random walk.



- The solution of the RW is *P(x,t)*,

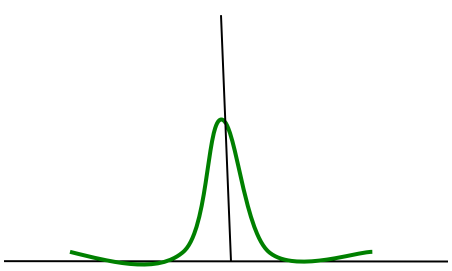
the probability for a particle to be

at position *x* at time *t*, how to determine it ?

- Problem treated by
- 1900: Bachelier (PhD student of Poincare), modelling of stock market temporal evolution.
- 1905: Einstein, modelling of Brownian motion.

Einstein’s formalism

- Assume RW in 1-D position space
- Introduce time interval Δt fixed,
  Δt << observation time,
  Δt > typical interaction time for a grain fluid-molecule collision
- The dust grain makes individual and subsequent jumps Δx,
  the Δx follow a certain probability distribution $q(\Delta x)$
  (i.e. the prob. for a jump Δx (with uncertainty $d\Delta x$) is $q(\Delta x)\, d\Delta x$)
- $q(\Delta x)$ is normalized, $s\; q(\Delta x)\; d\Delta x = 1$
  and let it be symmetric, for simplicity ($q(-\Delta x)=q(\Delta x)$)
- the dust grain makes only small jumps:
  $q(\Delta x)$ is non-zero only for small Δx
  (peaked and narrow)



Einstein's formalism, cont.

- We need to calculate *P(x,t),* the prob. for a particle to be at *x* at time *t*
- Assume we knew *P(x, t-Δt)* at an earlier time t-Δt, then

$$P(x,t) = P(x-\Delta x, t-\Delta t)\, q(\Delta x)$$

the prob. to be at *x* at time *t* equals
the prob. to have been at *x-Δx* at time *t - Δt* ago and
to have made a jump *Δx* in time *Δt*

- we still must sum over all possible *Δx*,

$$P(x,t) = \int_{-\infty}^{\infty} P(x-\Delta x, t-\Delta t)\, q(\Delta x)\, d\Delta x$$

RW equation in 1-D → integral equation, to be solved for unknown *P(x,t)*

Einstein's solution for *P(x,t)*

- Einstein-Bachelier equation

$$P(x,t) = \int_{-\infty}^{\infty} P(x-\Delta x, t-\Delta t)\, q(\Delta x)\, d\Delta x$$

- Only small jumps: *q(Δx)* non-zero only for small *Δx*, also *Δt* is small) Taylor expand *P(x-Δx, t-Δt)*,

$$P(x-\Delta x, t-\Delta t) = P(x,t) \; - \; \Delta t\, \partial_t P(x,t) + \ldots$$
$$- \; \Delta x\, \partial_x P(x,t) + \frac{1}{2} \Delta x^2\, \partial_x^2 P(x,t) + \ldots$$

- Insert

$$P(x,t) = \int P(x,t)\, q(\Delta x)\, d\Delta x \; - \; \int \Delta t\, \partial_t P(x,t)\, q(\Delta x)\, d\Delta x$$
$$- \; \int \Delta x\, \partial_x P(x,t)\, q(\Delta x)\, d\Delta x$$
$$+ \; \frac{1}{2} \int \Delta x^2\, \partial_x^2 P(x,t)\, q(\Delta x)\, d\Delta x$$

- Simplify

$$P(x,t) = P(x,t) \; - \; \Delta t\, \partial_t P(x,t)$$
$$+ \; \frac{1}{2} \sigma_{\Delta x}^2\, \partial_x^2 P(x,t)$$

- Simple diffusion equation !

$$\partial_t P(x,t) = \frac{\sigma_{\Delta x}^2}{2\Delta t} \partial_x^2 P(x,t)$$

Einstein's solution, cont.

- Integral equation turned to simple diffusion equation

$$\partial_t P(x,t) = \frac{\sigma^2_{\Delta x}}{2\Delta t} \partial_x^2 P(x,t)$$

with diffusion constant $D = \frac{\sigma^2_{\Delta x}}{2\Delta t}$

- In infinite system, when particles all start at *x=0* (*P(x,0)=δ(x)*) solution is known,

$$P(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$$

i.e. Gaussian, with time dependent variance $\sigma^2 = 2Dt$

- Mean square displacement:

$$\langle x^2(t) \rangle = \int x^2 P(x,t) = 2Dt$$

(just the variance of the Gaussian, per definition)

) normal diffusion

Normal diffusion should be the usual case

- Consider definition of RW

$$x_n = \Delta x_n + \Delta x_{n-1} + \Delta x_{n-2} + \ \dots \ + \Delta x_1 + x_0$$

- Central Limit Theorem (CLT) of probability theory:

  if all increments Δx_i
  - have finite mean μ and variance σ^2
  - are mutually independent
  - and their number is large

  then x_n has Gaussian distribution (here $\mu=0$, $x_0=0$),

$$P(x, t_n) = \frac{1}{\sqrt{2\pi t_n \sigma^2/\Delta t}} e^{-\frac{x^2}{2t_n\sigma^2/\Delta t}}$$

- with variance $t_n\sigma^2/\Delta t$ ($n=t_n/\Delta t$)

  (of course the s $\langle x^2(t_n)\rangle = \int x^2 \, P(x,t_n)\, dx = \text{variance} = t_n\sigma^2/\Delta t$
- MSD: → prop. to t_n) diffusion always normal
- Assumptions of CLT somehow natural: normal diffusion should be the usual case !

Normal Diffusion

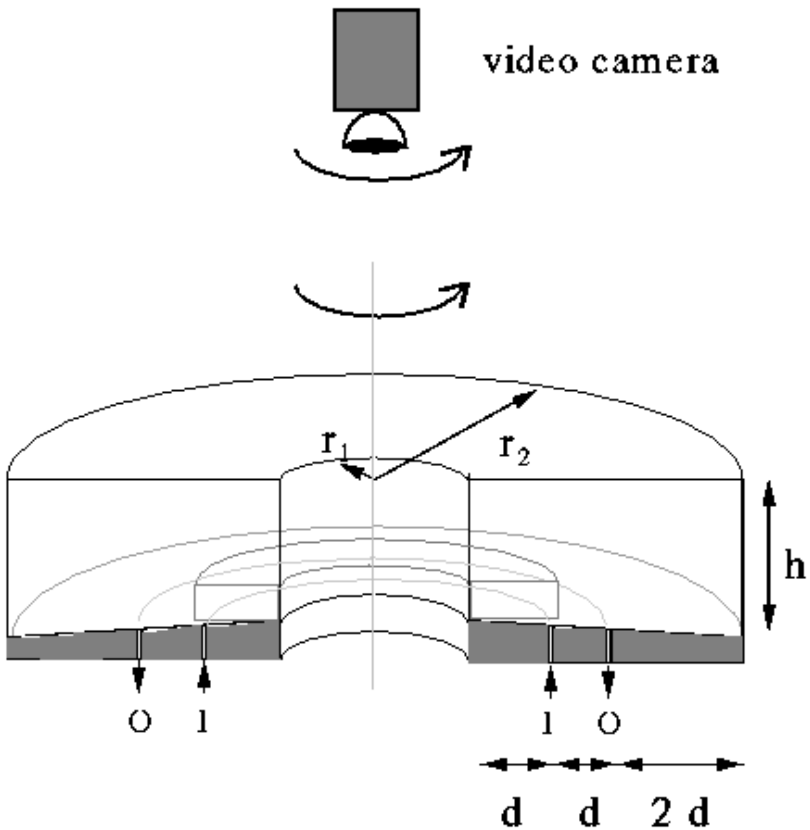
- 1. The mean square displacement

$$< r^2 >= Dt \quad or \quad D = \frac{< r^2 >}{t}$$

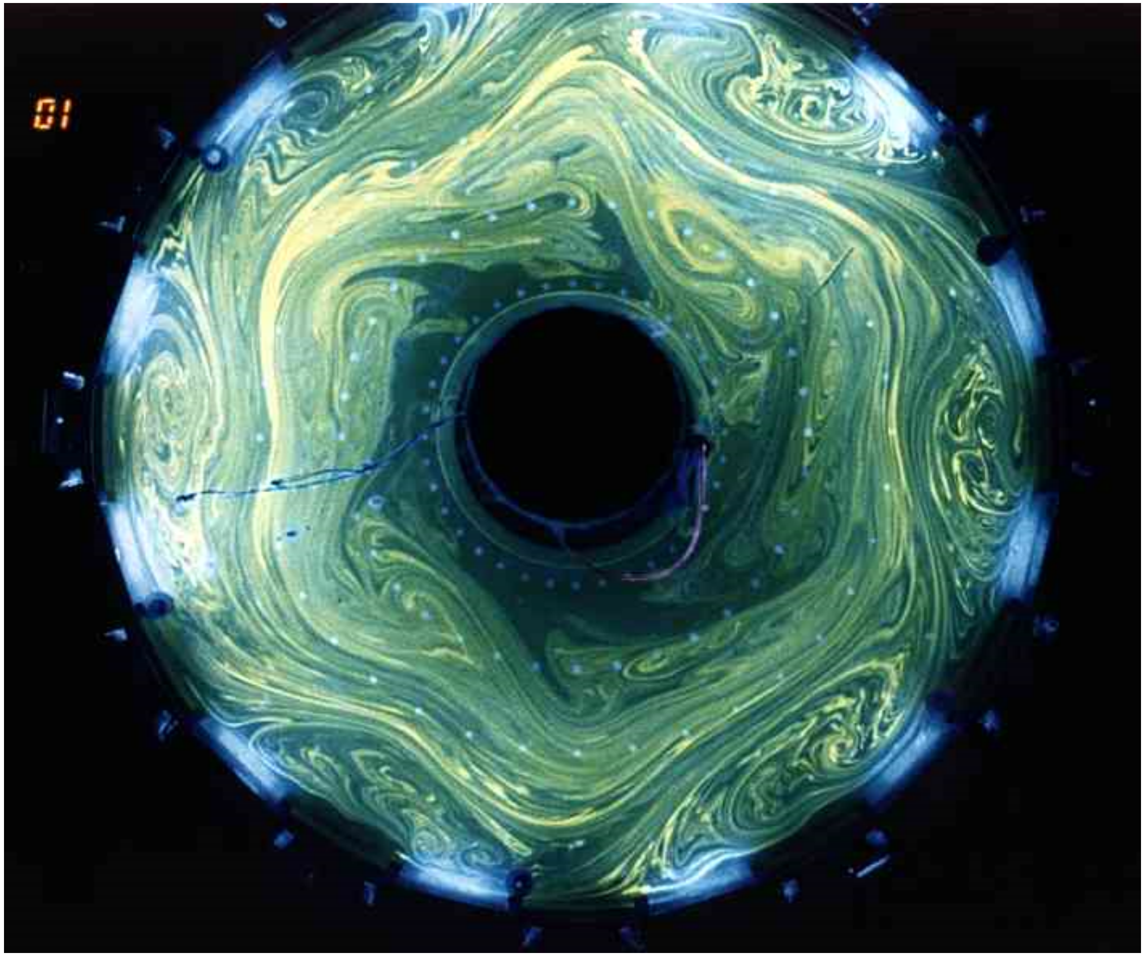
- 2. P(x,t)--Gaussian (normal) distributions .
- 3. Diffusion equation
- 4. Langevin's beautiful and simple formula can model the normal diffusion

Anomalous….Diffusion

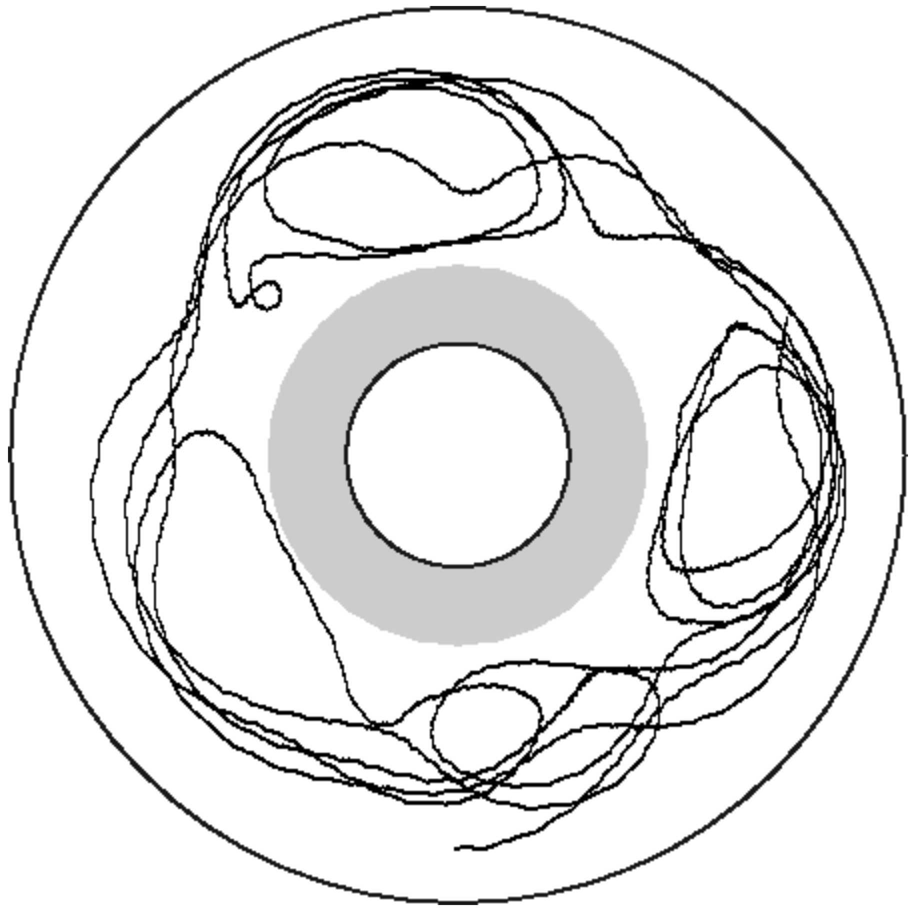
An experiment



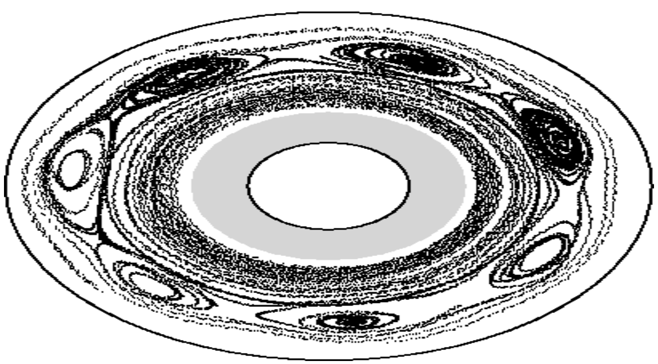
What we see

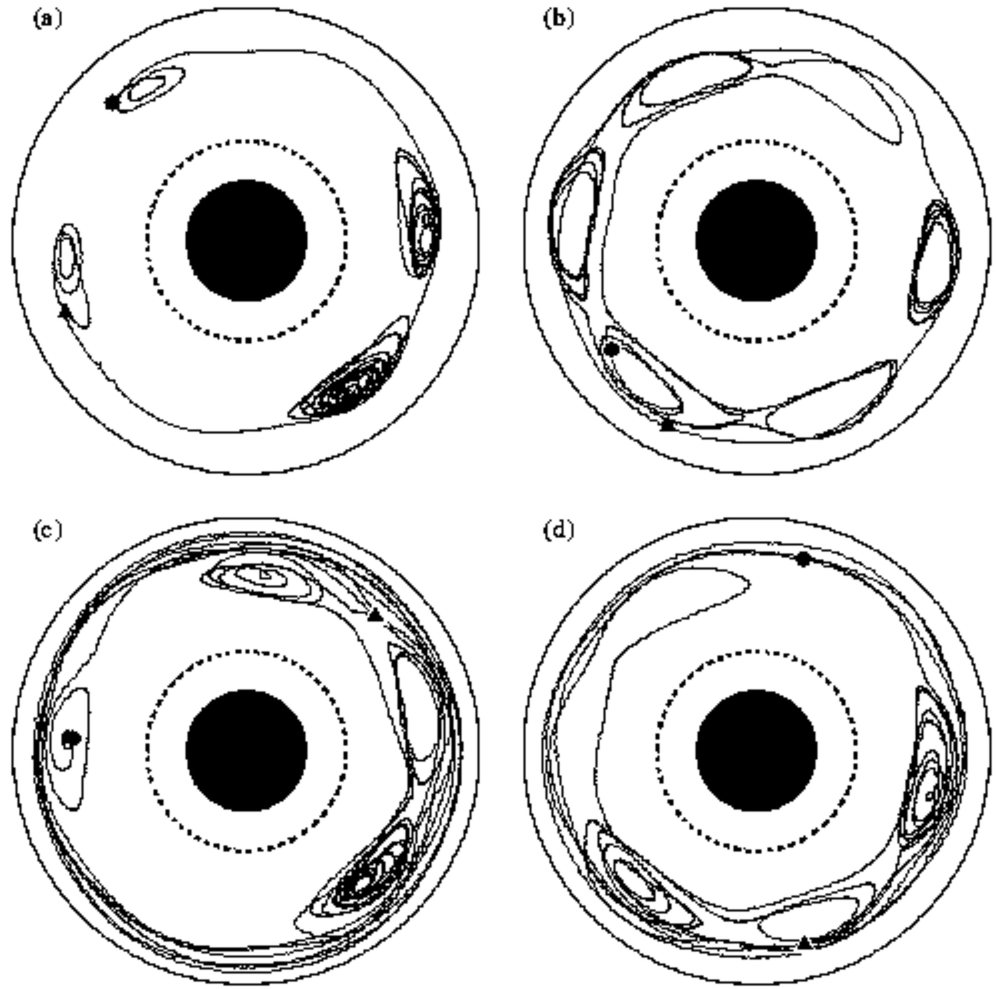


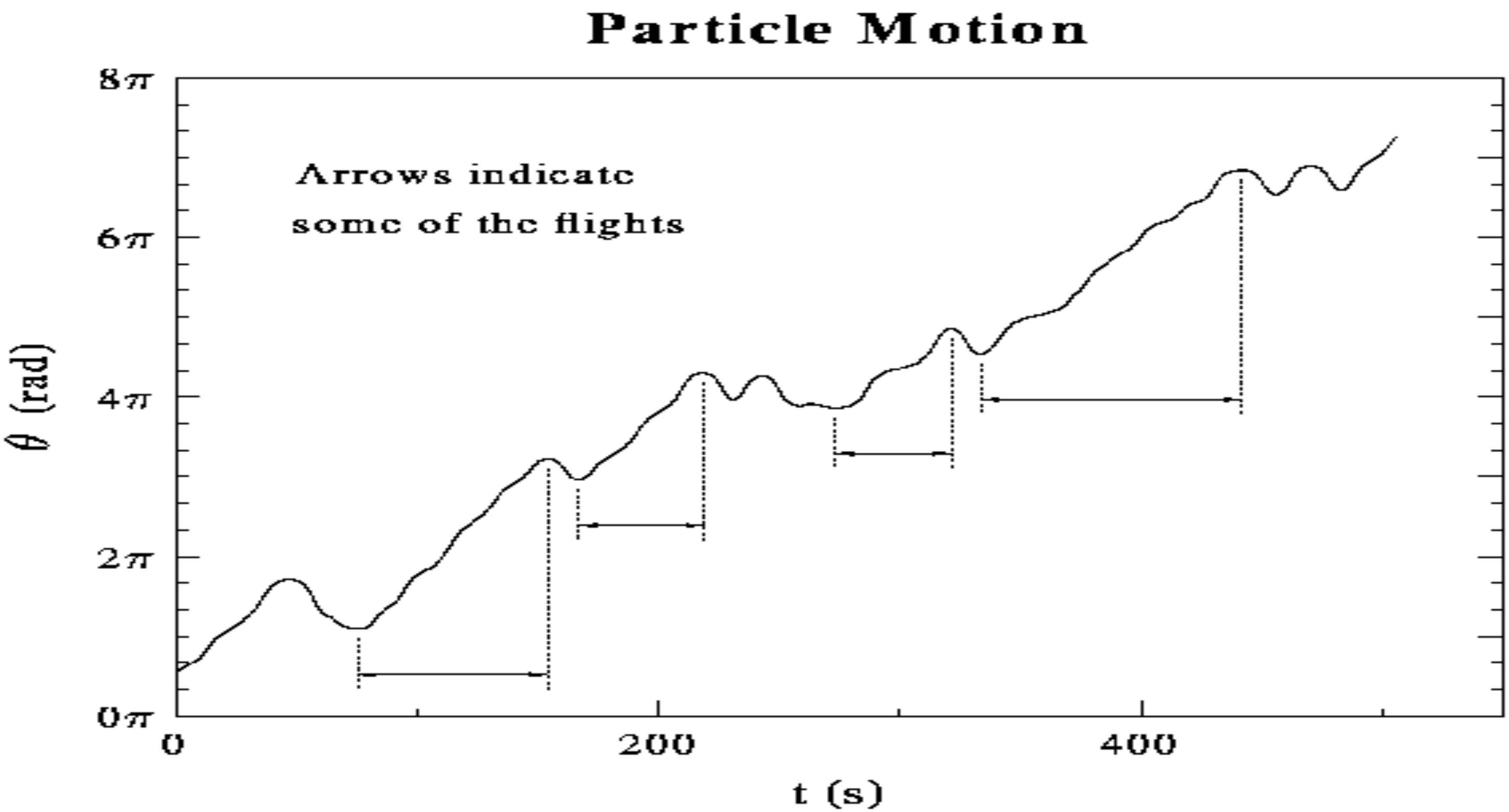
“Strange” walk



Trajectories inside the annulus

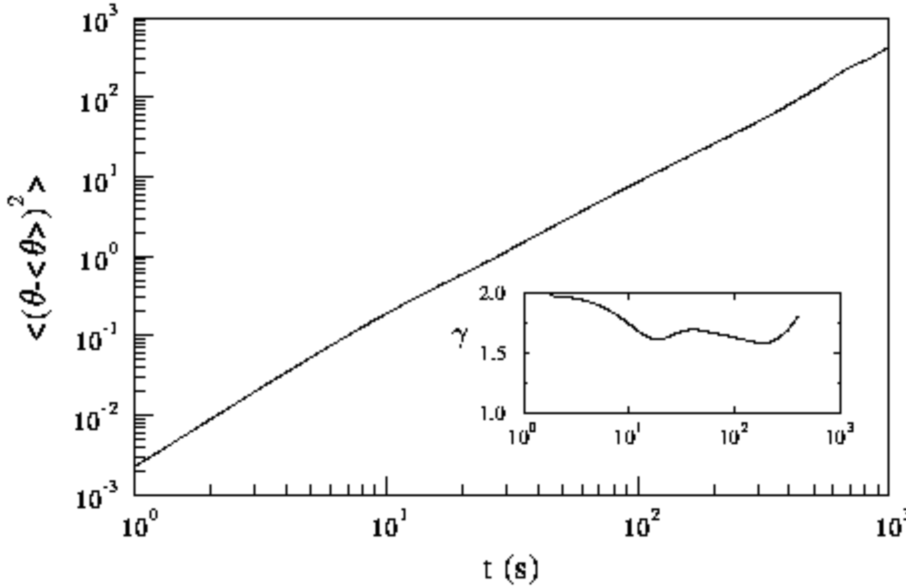




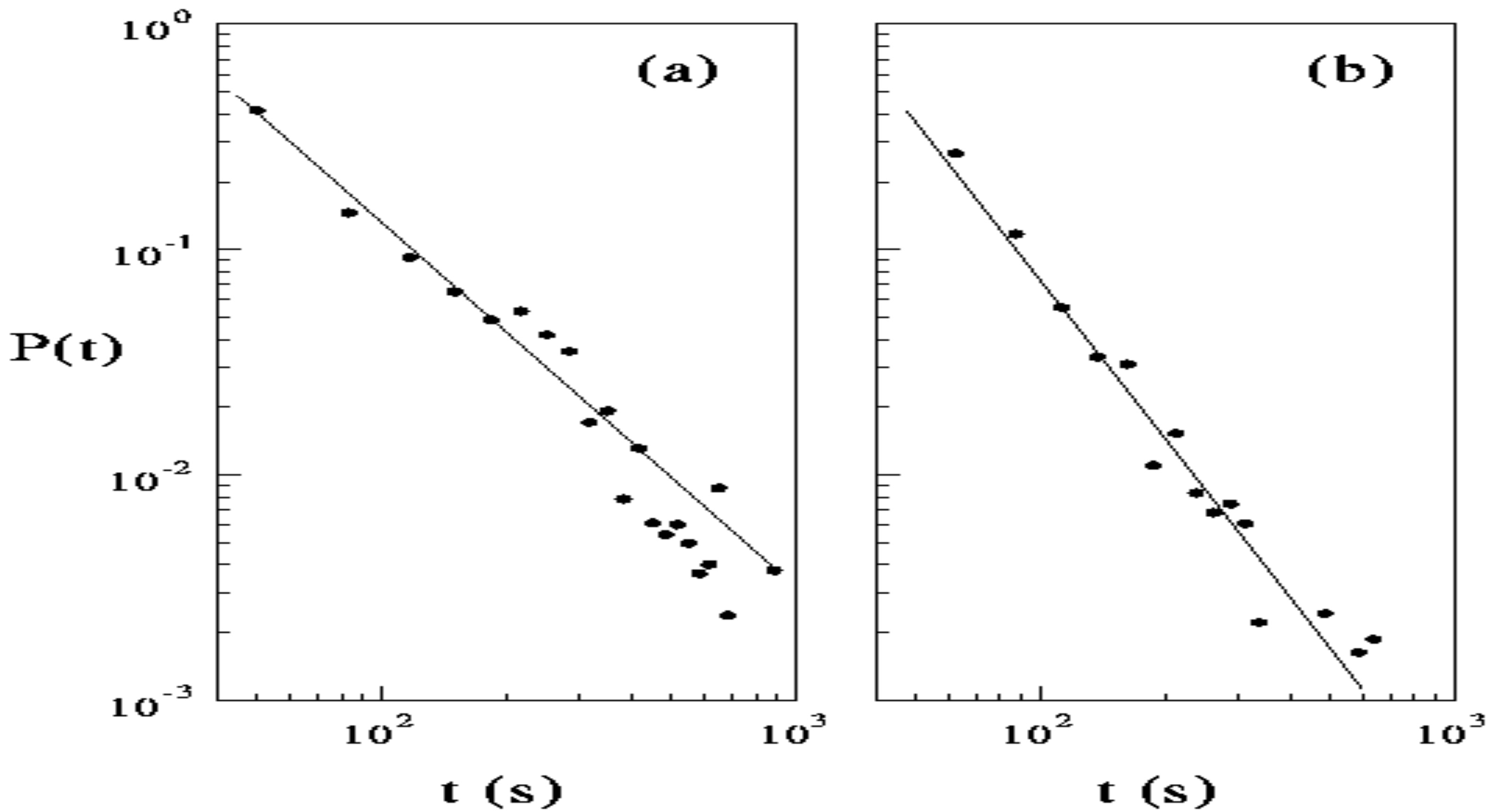


Particle Motion
8π
6π
4π
2π
0π
θ (rad)
Arrows indicatc
somc of thc flights
0
200
400
t (s)

$$< r^2 > \sim Dt^{1.6}$$

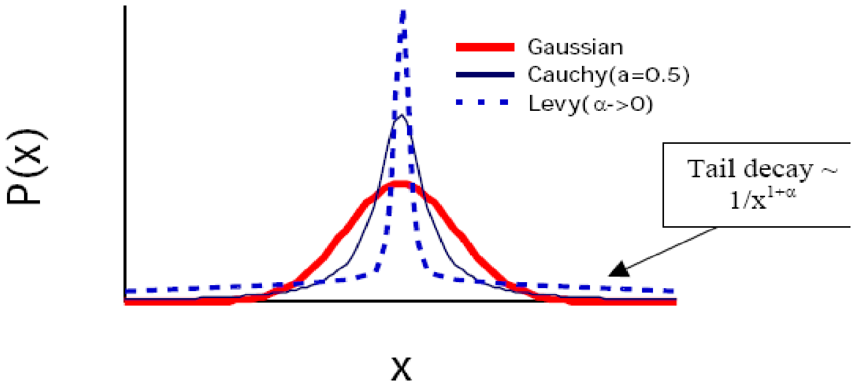


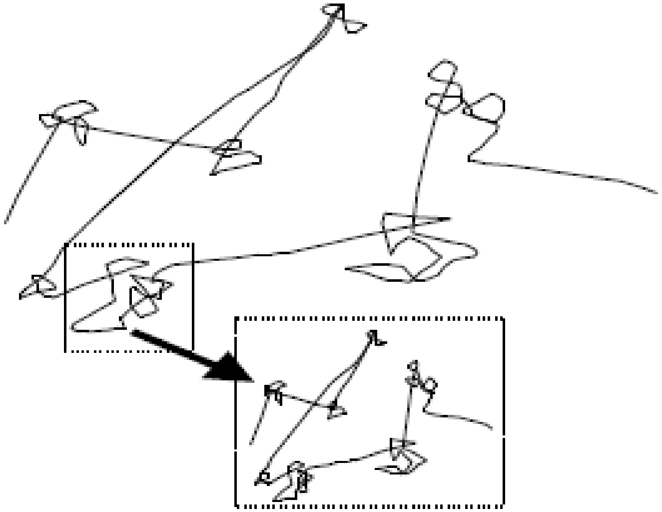
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
$<(\theta-<\theta>)^2>$
10^0
10^1
10^2
10^3
t (s)
2.0
1.5
1.0
γ



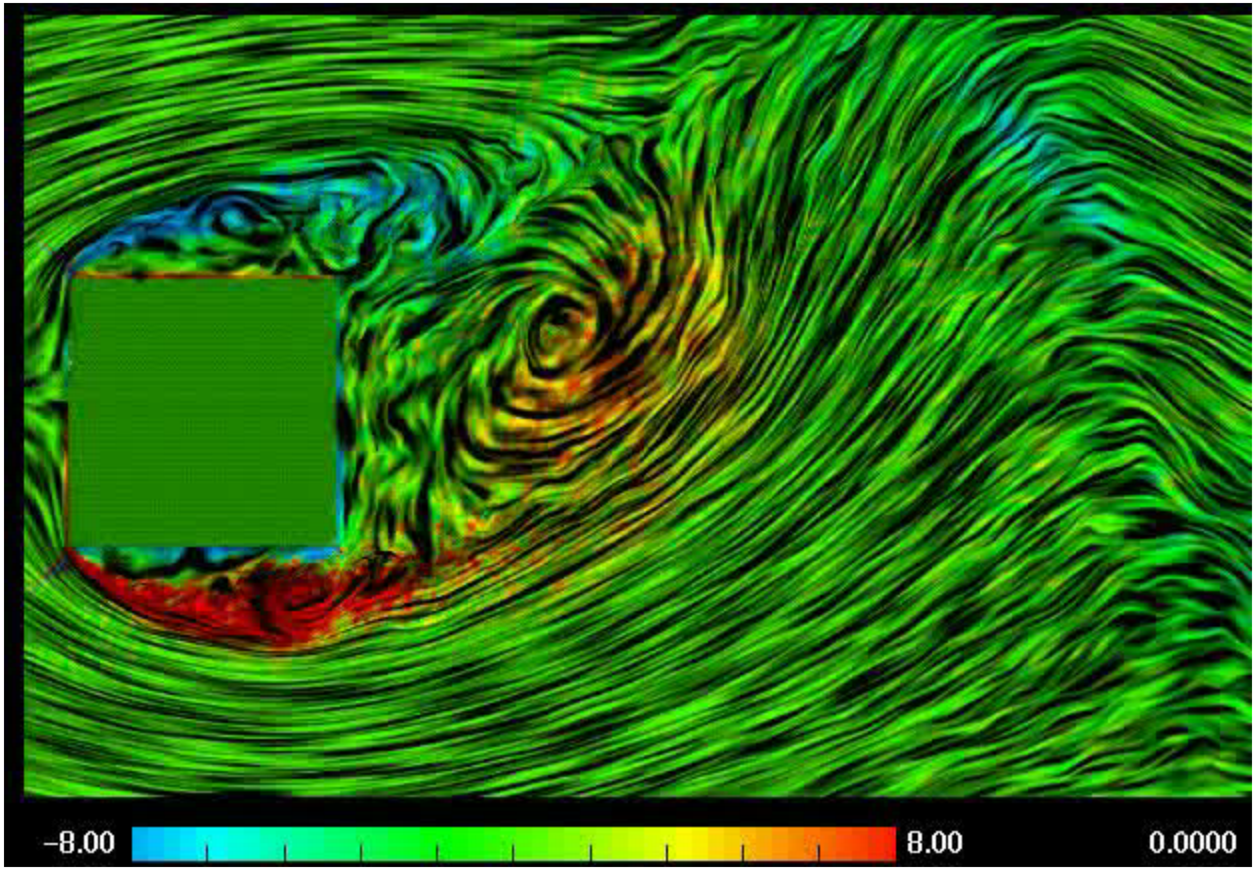
(a)
(b)
P(t)
10^0
10^{-1}
10^{-2}
10^{-3}
10^2
10^3
t (s)
t (s)

Levy walks and anomalous diffusion





Flow over an obstacle



Conditions for anomalous diffusion

- Central Limit Theorem (CLT) tries to make all diffusion normal
- For anomalous diffusion, we must violate at least one of its necessary conditions:
  (i) mean and/or variance of increments Δx_i must be infinite, or
  (ii) increments Δx_i must be mutually dependent, or
  (iii) the total number of increments must be small, or
  (iv) the time step Δt is not constant, anymore, but also random
- Point (iv) is what exactly is done in Continuous Time Random Walk (CTRW)
- Point (i) is realized by choice of particular distributions of increments $q(\Delta x)$, the Levy-distributions, with power-law tails:
  $q(\Delta x) \gg \Delta x^{-\alpha}$ (for Δx large)
  (Levy distributions make though sense only in the frame of CTRW)
- Point (iii): small number of steps: less than 30
  (with one step we would not talk about RW anymore)
- Point (ii) is an interesting possibility, could physically often be motivated – has it been tried ?

Continuous Time Random Walk (CTRW)

- Only position is considered (as in classical RW)
- Position of particle after *n*-steps (at time t_n): x_n

$$x_n = \Delta x_n + \Delta x_{n-1} + \Delta x_{n-2} + \ \ldots \ + \Delta x_1 + x_0$$

Δx_i: jump increment; x_0: initial position
→ as in classical RW

- New in CTRW: time t_n after *n* steps:

$$t_n = \Delta t_n + \Delta t_{n-1} + \Delta t_{n-2} + \ \ldots \ + \Delta t_1$$

Δt_i : time needed to perform *i*th step: now random
) also t_n random

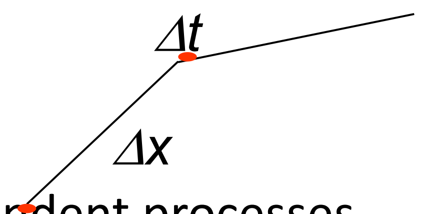
- Need now to specify distribution of jump increments Δx and of temporal increments Δt :
$q(\Delta x,\Delta t)$: probability to make a jump Δx and to spend a time Δt in the jump
- → RW completely specified:
problem: determine solution, i.e. prob. $P(x,t)$ that particle is at position x at time t

The distribution of increments

- General form $q(\Delta x, \Delta t)$: joint pdf, that specifies both the distribution of Δx and Δt (Δx and Δt might be mutually dependent)
- In practice, two cases are important and were investigated so far, related to different interpretation of what Δt represents:

1. consider Δt to be a waiting or trapping time

$$q(\Delta x, \Delta t) = q(\Delta x)\, q(\Delta t)$$



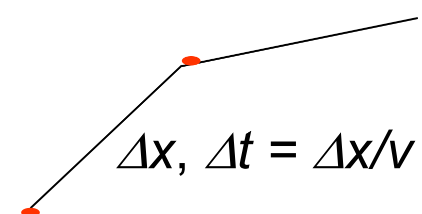
Δx and Δt are independent:
- waiting/being trapped and spatial jumping are independent processes
- $q(\Delta x)$ and $q(\Delta t)$ can be specified independently of each other

1. Consider Δt to be the time spent in the spatial increment:
assume a constant velocity v, then

$$\Delta t = \Delta x / v$$

i.e. Δt is given by Δx and v, and

$$q(\Delta x, \Delta t) = \delta(\Delta t - \Delta x/v)\, q(\Delta x)$$



The distribution of increments, cont.

- ‘Waiting/trapping model’:
  increments $q(\Delta x)\, q(\Delta t)$
  first version of CTRW historically introduced (1965, Montroll & Weiss)
  most published investigations/applications easiest to treat mathematically
  can though model only sub-diffusion not useful for our intended applications in confined plasma

$$\langle x^2(t)\rangle \propto t^\gamma, \text{ with } \gamma < 1$$

- ‘velocity model’:
  increments $\delta(\Delta t - \Delta x/v)\, q(\Delta x)$
  (introduced by Shlesinger & Klafter 1989) can model super-diffusion
  we focus mostly on the velocity model, in the following

$$\langle x^2(t)\rangle \propto t^\gamma, \text{ with } \gamma > 1$$

The CTRW equation I

- To treat the CTRW analytically, we need to derive its equation:

  waiting/trapping model: equation introduced in 1965 by Montroll and Weiss

  velocity model: equation introduced in 1989 by Shlesinger & Klafter
- Basically: generalize the Bachelier-Einstein equation

$$P(x,t) = \int_{-\infty}^{\infty} P(x - \Delta x, t - \Delta t)\, q(\Delta x)\, d\Delta x$$

- [Still, the equation must determine the probability *P(x,t)* for a particle to be at position *x* at time *t*]
- [CTRW can also be implemented numerically as a Monte-Carlo simulation: let the computer trace the particles which make their random jumps]

The CTRW equation II

- Generalize the Bachelier Einstein equation

$$P(x,t) = \int_{-\infty}^{\infty} P(x - \Delta x, t - \Delta t)\, q(\Delta x)\, d\Delta x$$

- New symbol *Q*. Idea: connect *Q(x,t)* to *Q(x-Δ x,t-Δ t)* in the past:

Q(x,t) = Q(x-Δx, t-Δt) £ prob. to make a jump *Δx* in time *Δt*

i.e.

Q(x,t) = Q(x-Δx, t-Δt) £ δ(Δt - Δx/v) q(Δx)

Prob. to be at *x* at time *t* equals probability to have been at time *t - Δt* at position *x - Δx*, and to have made a spatial jump *Δx* that took a time *Δt*

- Still need to sum over all possible *Δx, Δt*

$$Q(x,t) = \int dx \int_0^t dt\, Q(x-\Delta x, t-\Delta t)\, \delta(\Delta t-\Delta x/v)\, q(\Delta x)$$

very close generalization of Bachelier-Einstein

Introduction I

- Particle transport in weakly turbulent environments ($\delta B/B << 1$) has been discussed extensively with the use of the Fokker-Planck (FP) equation, mostly in combination with the quasi-linear (QL) approximation

Strong turbulence is though also important and abundant

Recent research on the development of strong magnetic turbulence ($\delta B/B \approx 1$) has shown the importance of two scenarios:

1. Extended current filaments (CF) or multiple interacting CFs develop on fast time scales into a strongly turbulent environment, fragmented into a collection of small scale CFs.
2. Propagating Alfvén waves reinforce reconnection at existing CF and new CF are formed.

Introduction II

In this context, we address two open questions:

1. Is the FP equation still valid in strongly turbulent environments ?
2. How to model transport when the FP approach is not valid anymore ?

- In the following
  1. we consider a large scale environment of strong turbulence
  2. and we analyze statistically the energization of particles in this environment, focusing on the high energy part (tail) of the energy distribution.
  3. we develop an appropriate transport model
- Applications: Solar flares, Earth's magnetosphere, accretion disks, jets, ..., may-be the plasma edge in tokamaks ?

The MHD turbulent environment I

- We consider a strongly turbulent environment as it naturally results from the nonlinear evolution of the MHD equations
- We do not set up a specific reconnection geometry
- 3D, nonlinear, resistive, compressible and normalized MHD equations

$$\partial_t \rho = -\nabla \cdot \mathbf{p} \quad (1)$$

$$\partial_t \mathbf{p} = -\nabla \cdot (\mathbf{pu} - \mathbf{BB}) - \nabla P - \nabla B^2/2 \quad (2)$$

$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E} \quad (3)$$

$$\partial_t (S\rho) = -\nabla \cdot [S\rho\mathbf{u}] \quad (4)$$

with ρ the density, $\mathbf{p}$ the momentum density, $\mathbf{u} = \mathbf{p}/\rho$, P the thermal pressure, $\mathbf{B}$ the magnetic field, $\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \eta\mathbf{J}$ the electric field, $\mathbf{J} = \nabla \times \mathbf{B}$ the current density, η the resistivity, $S = P/\rho^\Gamma$ the entropy, and $\Gamma = 5/3$ the adiabatic index.

- The MHD equations are solved numerically with the pseudo-spectral method combined a the strong-stability-preserving Runge Kutta scheme of
  - Cartesian coordinates
  - periodic boundary conditions

The MHD turbulent environment II

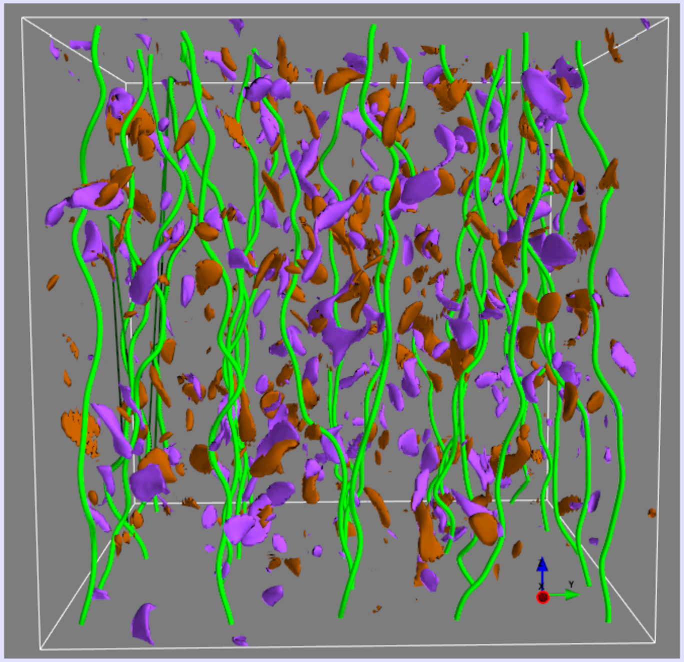
- initial conditions: superposition of Alfvén waves, with a Kolmogorov type spectrum
- constant background magnetic field B_0 in the z-direction.

- The mean value of the initial magnetic perturbation is $< b >= 0.6B_0$, its standard deviation is $0.3B_0$, so that we indeed consider strong turbulence.
- For the MHD turbulent environment to build, we let the MHD equations evolve until the largest velocity component starts to exceed twice the Alvfèn speed.
- The magnetic Reynolds number at final time is $< |\mathbf{u}| > l/\eta = 3.5 \times 10^3$
- The test-particle are tracked in a fixed snapshot of the MHD evolution
- Also, we take into account anomalous resistivity effects by increasing the resistivity to $\eta_{an} = 1000\eta$ locally when the current density $J = |\mathbf{J}|$ exceeds a threshold J_{cr}.

The MHD turbulent environment III

Iso-contours of the supercritical current density component J_z (positive in brown, negative in violet), magnetic field lines (green):

clear fragmentation into a large number of small-scale coherent structures



Test-particle simulations I

- The relativistic guiding center equations (without collisions) are used for the evolution of the position $\mathbf{r}$ and the parallel component $u_{||}$ of the relativistic 4-velocity of the particles (X. Tao et al., PoP **14**, 092107 (2007))

$$\frac{d\mathbf{r}}{dt} = \frac{1}{B^*_{||}} \left[\frac{u_{||}}{\gamma} \mathbf{B}^* + \hat{\mathbf{b}} \times \left(\frac{\mu}{q\gamma} \nabla B - \mathbf{E}^* \right) \right] \qquad (5)$$

$$\frac{du_{||}}{dt} = -\frac{q}{m_0 B^*_{||}} \mathbf{B}^* \cdot \left(\frac{\mu}{q\gamma} \nabla B - \mathbf{E}^* \right) \qquad (6)$$

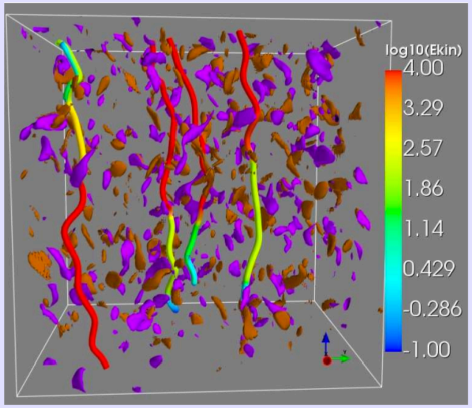
where $\mathbf{B}^* = \mathbf{B} + \frac{m_0}{q} u_{||} \nabla \times \hat{\mathbf{b}}$, $\mathbf{E}^* = \mathbf{E} - \frac{m_0}{q} u_{||} \frac{\partial \hat{\mathbf{b}}}{\partial t}$, $\mu = \frac{m_0 u_\perp^2}{2B}$ is the magnetic moment, $\gamma = \sqrt{1 + \frac{u^2}{c^2}}$, $B = |\mathbf{B}|$, $\hat{\mathbf{b}} = \mathbf{B}/B$, $u_\perp$ is the perpendicular component of the relativistic 4-velocity, and q, m_0 are the particle charge and rest-mass, respectively.

- The test-particles we consider throughout are electrons. Initially, all particles are located at random positions, they obey a Maxwellian distribution with temperature $T = 100\,\mathrm{eV}$. The simulation box is open, the particles can escape from it when they reach any of its boundaries.

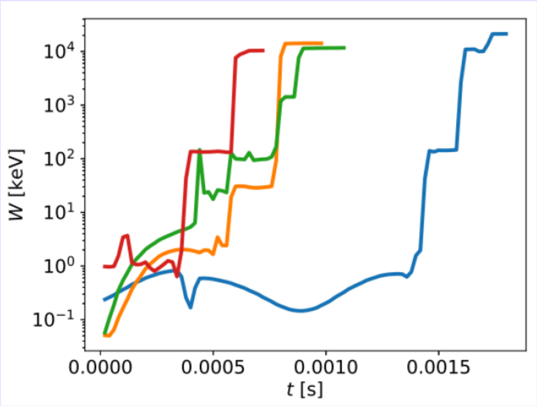
Test-particle simulations II

- The acceleration process, is very efficient, and we consider a final time of 0.002 s (7×10^5 gyration periods), at which the asymptotic state has already been reached.

4 orbits of energetic particles (reaching 10 MeV), colored according to the logarithm of their kinetic energy in keV



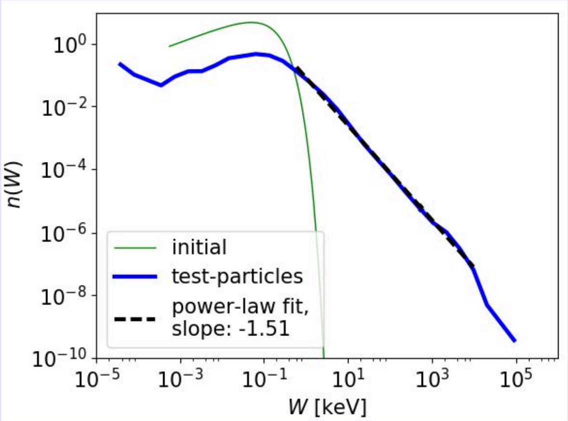
The energy evolution of the same 4 energetic particles



The particles mostly gain energy in a number of sudden jumps in energy,
the energization process thus is localized,
and there is multiple energization at different current filaments

Test-particle simulations III

the energy distribution at final time (blue):
clear power law tail,
power-law index -1.51



Transport coefficients and classical FP equation I

Question 1

Can the test-particle results be reproduced as a solution of the FP equation ?

- For simplification, we consider the FP equation only in energy space

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial W}\left[Fn - \frac{\partial [Dn]}{\partial W}\right] = -\frac{n}{t_{esc}}, \quad (7)$$

n: the distribution function, W: kinetic energy, t_{esc}: the escape time.
D is the energy diffusion coefficient,

$$D(W,t) = \frac{\left\langle (W(t+\Delta t) - W(t))^2 \right\rangle_W}{2\Delta t}, \quad (8)$$

F is the energy convection coefficient,

$$F(W,t) = \frac{\langle W(t+\Delta t) - W(t)\rangle_W}{\Delta t}, \quad (9)$$

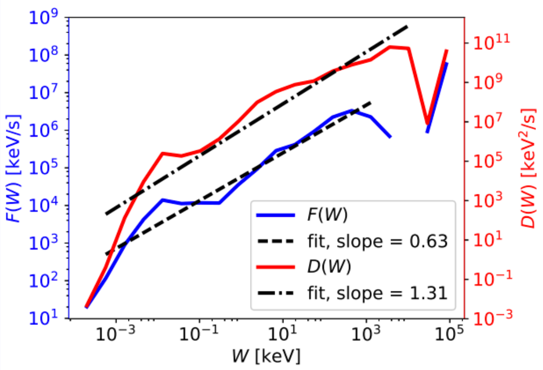
with Δt a small time-interval.
$\langle ...\rangle_W$ denotes the conditional average that $W(t) = W$

Transport coefficients and classical FP equation II

- For the estimate of the coefficients F, D from the simulation data: we monitor the particle energy at a number of fixed times separated by Δt, the conditional averaging is done through binned statistics
  - divide the energies of the particles at time t into a number of logarithmically equi-spaced bins and perform the requested averages separately for the particles in each bin.

The estimates of $F(W)$ and $D(W)$ at $t = 0.002\,s$ as function of the energy:

—> power-law shape, indices $a_F = 0.63$ and $a_D = 1.31$.



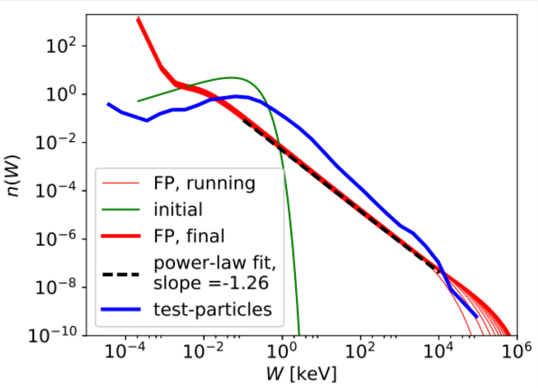
- Verification of the estimates of F and D: insert F and D, into the FP equation and solve it numerically in $[0, \infty)$ (pseudospectral method, based on rational Chebyshev polynomials)
  - escape time estimate $t_{esc} = 0.004\,s$ (assuming the number of particles staying in the box to decay exponentially)

Transport coefficients and classical FP equation III

The solution of the FP equation up to final time 0.002 s:

—> clear power-law tail,

—> much flatter though than the test-particle simulations.

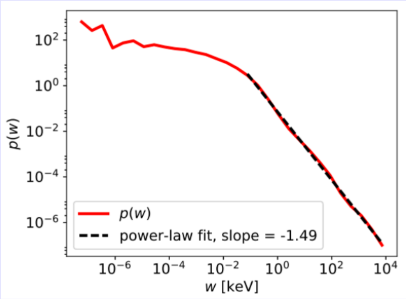


- In Vlahos et al., ApJ **827**, L3, (2016) we have shown that the above procedure can be successful:
  Why does it fail here ?

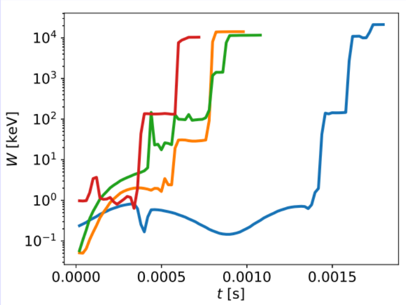
Transport coefficients and classical FP equation IV

estimates of F and D are based on the sample of energy increments
$w_j := W_j(t + \Delta t) - W_j(t)$
(with j the particle index)

The distribution of increments has a power law tail (index -1.49)



—> occasionally very large jumps in energy space: Levy flights



- energy increments with a power-law tail imply:
  1. The estimates of F as a mean value and D as a variance theoretically are infinite, and thus in practice they are very problematic
  2. The prerequisites for deriving a FP equation are not fulfilled (see below)

Fractional transport equation (FTE) I

Question 2

How to model transport when the FP approach is not valid anymore ?

- General description of transport in energy space: Chapman-Kolmogorov equation

$$n(W,t) = \int dw \int_0^t d\tau\, n(W-w, t-\tau)\, q_w(w)\, q_\tau(\tau) + n(W,0)\int_t^\infty q_\tau(\tau) d\tau \qquad (10)$$

  expresses a conservation law, and can be interpreted as a Continuous Time Random Walk.

- q_w: probability density for a particle to make a random walk step w in energy,
  q_τ: probability density for this step to be performed in a time interval τ
- When both q_w and q_τ have finite mean and variance (i.e. only small increments) (as e.g. for Gaussians), then the FP equation can be derived from Eq. (10) through Taylor-expansions

Fractional transport equation (FTE) II

- Here, we do not make the assumption of small increments
- distribution of increments, expressed in Fourier (k) and Laplace space (s):
  1. distribution of energy increments: symmetric stable Levy distributions
     $\hat{q}_w(k) = \exp(-a|k|^\alpha)$, with $0 < \alpha \leq 2$,
     which exhibit a power-law tail in energy-space, $q_w(w) \sim 1/w^{1+\alpha}$ for $\alpha < 2$ and w large,
     and for $\alpha = 2$ they are Gaussian distributions
  2. waiting time distribution: one sided stable Levy distributions,
     $\tilde{q}_\tau = \exp(-bs^\beta)$, with $b > 0$ and $0 < \beta \leq 1$,
     which have a power-law tail, $q_\tau \sim 1/\tau^{1+\beta}$ for $\beta < 1$ and τ large,
     and for $\beta = 1$ they equal $q_\tau(\tau) = \delta(\tau - b)$
- In order to derive a meso-scopic transport equation, we consider the fluid-limit: w, τ are large, and thus k, s are small, so that the distributions of increments can be approximated as
  $\hat{q}_w \approx 1 - a|k|^\alpha$
  $\tilde{q}_\tau \approx 1 - bs^\beta$.

Fractional transport equation (FTE) III

- Chapman Kolmogorov equation —> make Fourier Laplace transform —> apply convolution theorems —> insert distributions of increments in the fluid limit:

$$bs^{\beta}\tilde{\hat{n}}(k,s) - bs^{\beta-1}\hat{n}(k,0) = -a|k|^{\alpha}\tilde{\hat{n}}(k,s) \quad (11)$$

which can be written as a fractional transport equation (FTE)

$$bD_t^{\beta}n = aD_{|W|}^{\alpha}n \quad (12)$$

with D_t^{β} the Caputo fractional derivative of order β, defined in Laplace space as

$$\mathcal{L}\left(D_t^{\beta}n\right) = s^{\beta}\tilde{n}(W,s) - s^{\beta-1}n(W,0) \quad (13)$$

and $D_{|W|}^{\alpha}$ the symmetric Riesz fractional derivative of order α, defined in Fourier space as

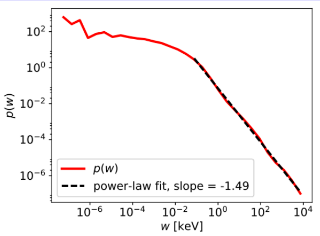
$$\mathcal{F}\left(D_{|W|}^{\alpha}n\right) = -|k|^{\alpha}\hat{n}(k,t) \quad (14)$$

- We need to estimate two parameter sets, α, a and β, b

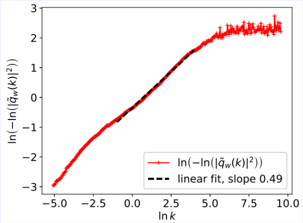
Fractional transport equation (FTE) IV

- the order of the fractional derivatives (α, β) is given by the index of the power-law tail of the distribution of increments, if any
  - otherwise, if the mean and variance of the increments are finite, then the classical FP equation is appropriate.

the distribution of energy increments $p_w(w)$ has a power-law tail, its index z yields $\alpha = -z - 1 = 0.49$.



- As second method to determine α and also a, we use the characteristic function approach:
  $\alpha = 0.49$ (as before) and $a = 0.36$



Fractional transport equation (FTE) V

- "Waiting times": We have considered energy increments over a fixed time interval Δt,
  —> we use 'observation/sampling times', not 'waiting times'
  —> "waiting time" distribution $p_\tau(\tau) = \delta(t - \Delta t)$,
  —> it follows that $\beta = 1$ and $b = \Delta t$.
    - This approach seems unavoidable if the test-particle data are given in the form of time-series, where there is no direct information on the waiting times between scattering events.
- Thus, we consider the fractional transport equation to have a first order derivative in time-direction and a fractional derivative in energy direction,

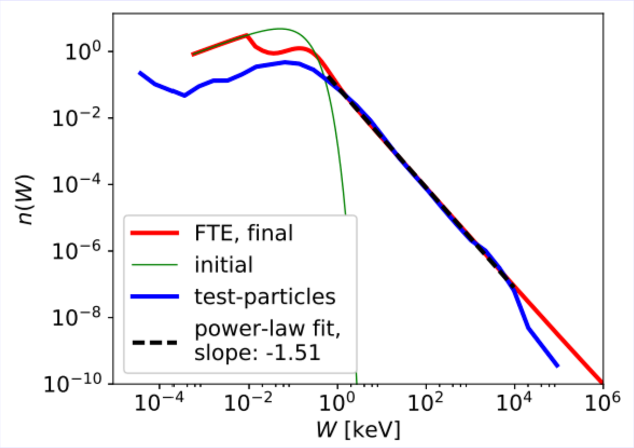
$$\partial_t n = (a/b) D^{\alpha}_{|W|} n - n/t_{esc}, \qquad (15)$$

where we also have added an escape term.

Fractional transport equation (FTE) VI

- numerical solution of the FTE:
  Grünwald-Letnikov definition of fractional derivatives (e.g. (Kilbas et al.(2006))), in the matrix formulation of (Podlubny et al.(2009), Podlubny et al.(2013)):
  same non equi-distant grid-points in $[0, \infty)$ as above for the FP equation

Solution of the FTE at $t = 0.002$ s:
the FTE reproduces very well the power-law tail from the test-particle simulations in its entire extent



Conclusion I

We posed two questions:

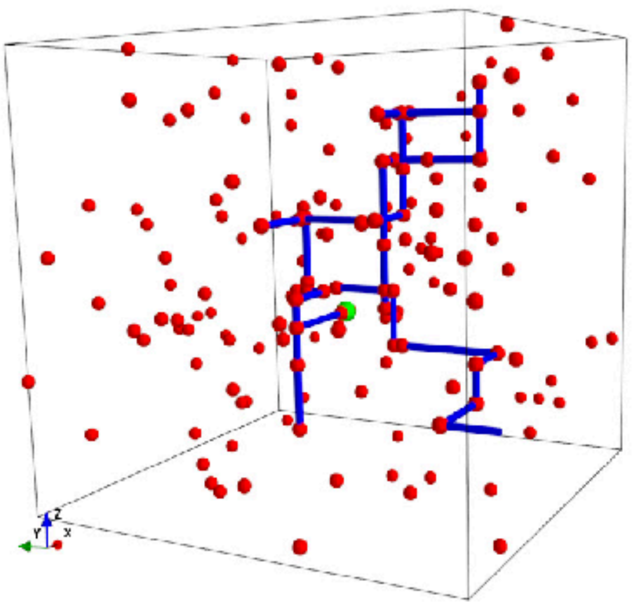
1. Is the FP equation still valid in strongly turbulent environments ?
   Answer: No !
2. How to model transport when the FP approach is not valid anymore ?
   Answer: With a kind of fractional transport equation (work is still needed)

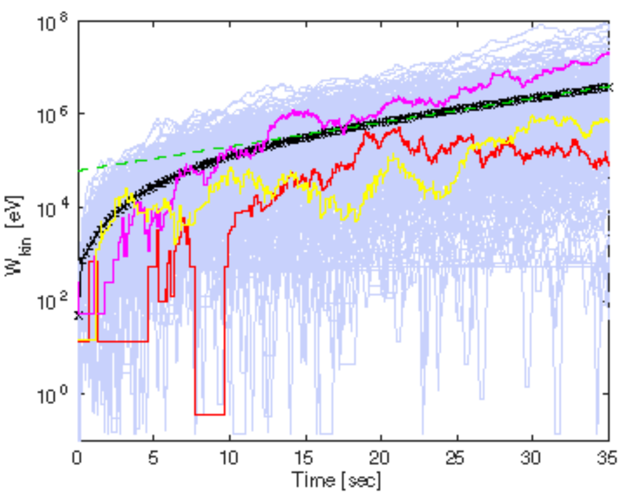
- statistical analysis of the distribution of energy increments:
  —> allows deciding whether a FP or a FTE is appropriate
  —> in the FP case the estimate of the transport coefficients is based on it
  —> In the FTE case, the form of the FTE and its parameters (the order of the fractional derivative etc), are directly inferred from the simulation data (and thus they are not universal or unique).
- simplifying assumption:
  instead of 'waiting times' we used 'observation/sampling times'
  —> did not affect the success of the FTE approach
- We made no effort to model the low energy part of the distribution
- published in Phys. Rev. Lett. **119**, 045101 (2017)

How far can we go by using the initial ideas of Fermi

The strong turbulent environment can be modeled in line with the initial idea of Fermi, where the strong scatterers can replace the "magnetic clouds" and the particles gain energy stochastically (second order Fermi) Pisokas et al. 2017 [Pisokas et al. (2017)]

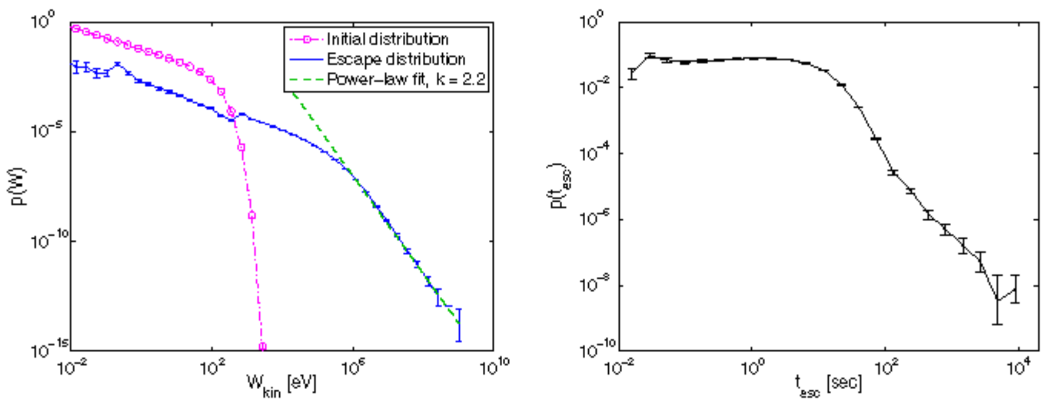
$$\Delta W = \frac{2}{c^2}(V_A^2 - \vec{V}_A \cdot \vec{u})W$$





How far can we go by using the initial ideas of Fermi

Using typical parameters, length of the box $L = 10^{10}cm$, density $n = 10^9 cm^{-3}$ magnetic field 100 Gauss, the temperature $T = 100eV$ and characteristic distance between the scatterers $\lambda_{sc} \approx 10^8 cm$ the characteristic time for the particles to reach an asymptotic state will be $\approx 10 secs$. The plasma will be heated and the tail has been accelerated to very high energies stochastically. The mean escape time from the box is approximately equal with the acceleration time (around 10 secs) and the power law reaches asymptotically an index around 2!



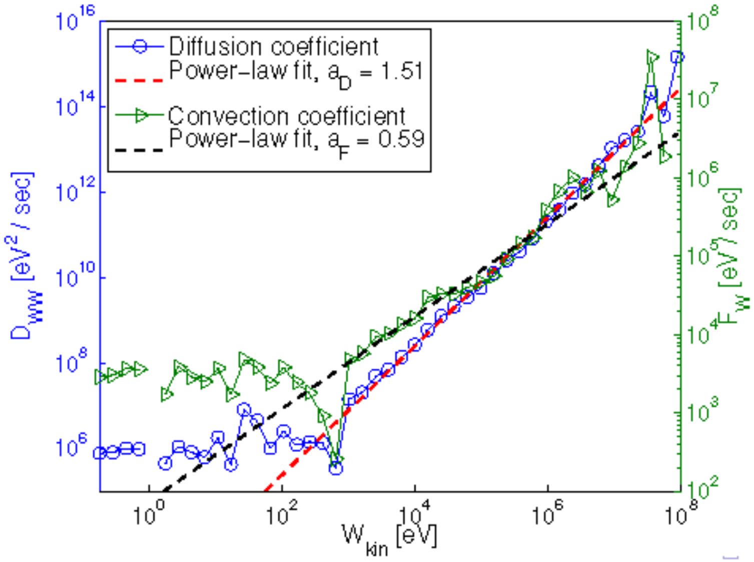
Fermi predicted that $F(W) \approx W^{-k}$ $k = 1 + t_{acc}/t_{esc}$

Energy transport for the stochastic Fermi acceleration

How we will estimate the transport coefficients?
We use the orbits and the formulae

$$D_{WW}(W) = \frac{< (W(t+\Delta t) - W(t))^2 >_W}{2\Delta t}$$

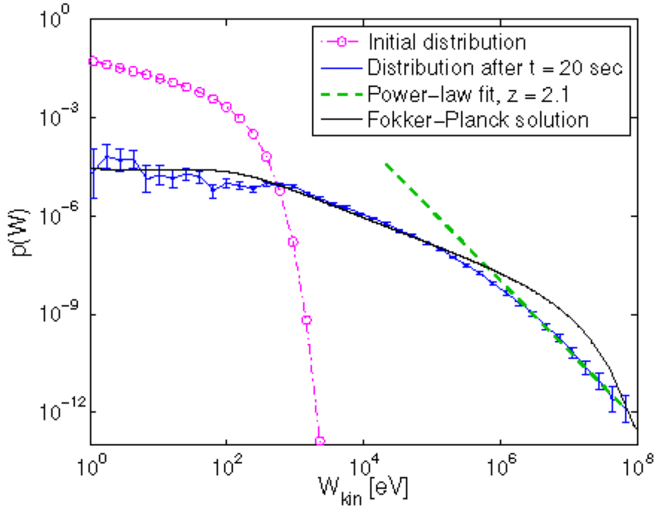
$$F_W(W) = \frac{< W(t+\Delta t) - W(t) >_W}{\Delta t}$$



Energy transport for the stochastic Fermi acceleration

Using the transport coefficients and the estimated mean escape time we solve the transport equation

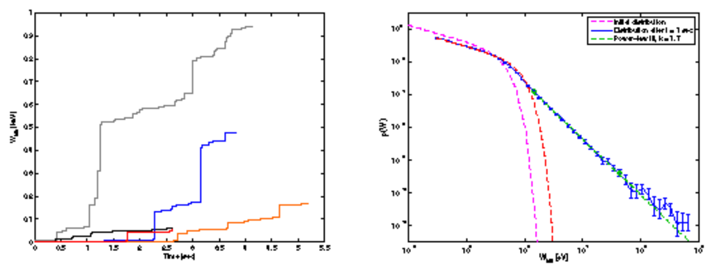
$$\frac{\partial f(W,t)}{\partial t} = \frac{\partial [F_W(W) f(W,t)]}{\partial W} + \frac{\partial^2 [D_{WW}(W) f(W,t)]}{\partial W^2} - \frac{f(W,t)}{t_{esc}} \quad (15)$$



How far can we go by using the initial ideas of Fermi

Vlahos et al. Apj Letters 2016, Isliker et al. Ap. J. 2017
[Vlahos et al. (2016), Isliker et al. (2017)]
The strong turbulent environment can be modeled in line with the initial idea of Fermi, where the Reconnecting Current Sheets can replace the "magnetic clouds" and the particles gain energy systematically (First order Fermi). The crucial parameter here is the mean distance of the scatterers λ_{sc}

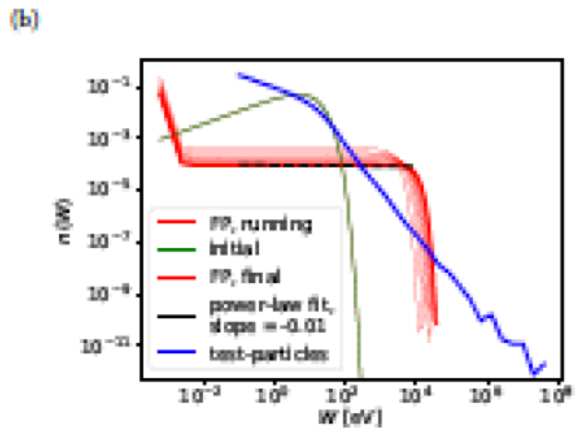
$$\Delta W = | q | E_{eff} \ell_{eff}$$



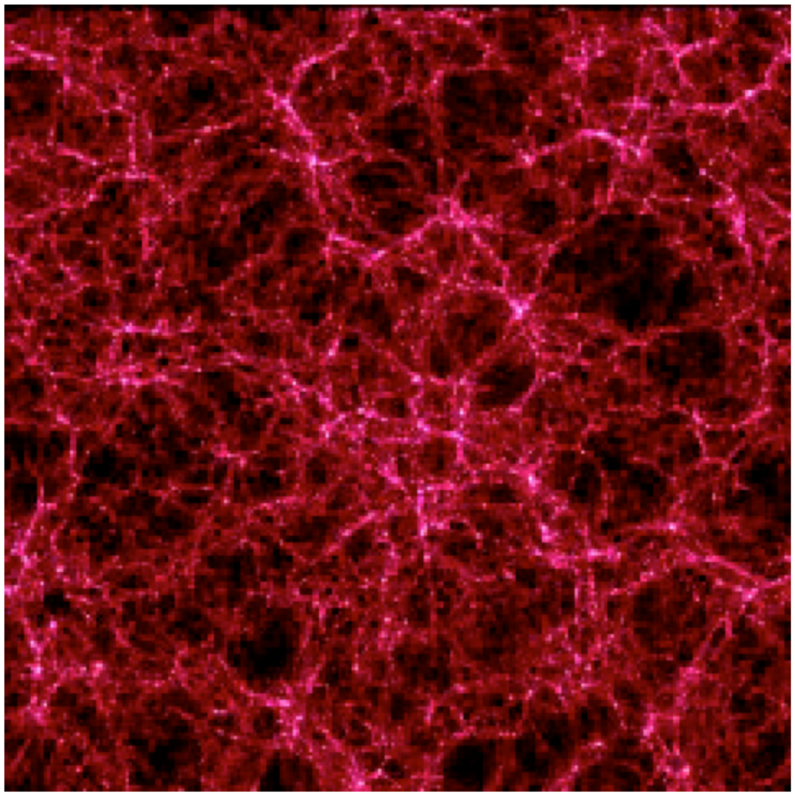
The most important finding in this study is the failure of the FP equation to reproduce the test particle results of a systematic accelerator

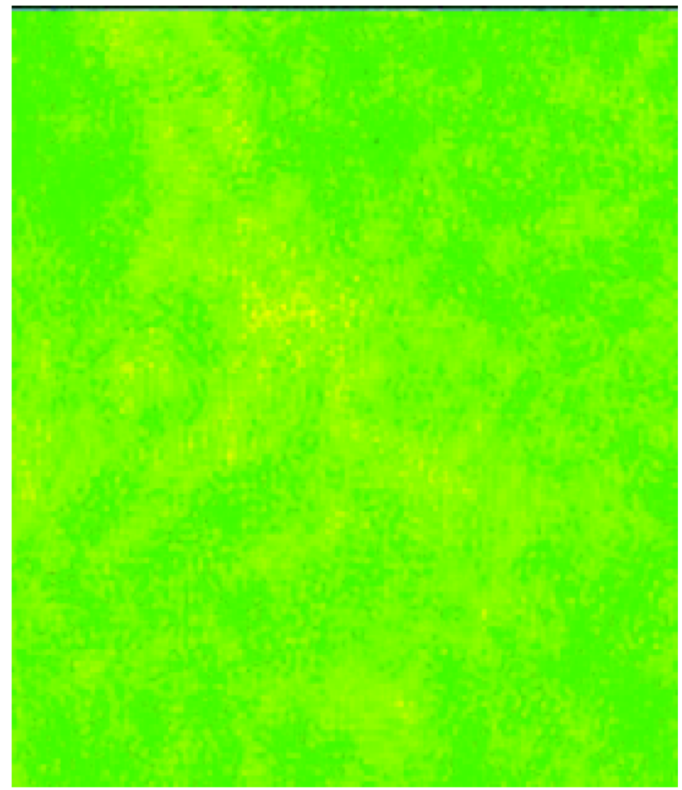
Failure of Fokker-Planck equation

The failure of the solution of the FP equation to represent the results from the test particle simulation



An open problem….Diffusion of Cosmic Rays through a fractal Universe: Moving through voids and localized action on gallaxies





Conclusions

- We have discussed the importance of random walk in nature and its relation to normal diffusion in stable systems.
- We have discussed a prototype of stochastic differential equations-The Langevin equation.
- We introduced the notions of anomalous diffusion-Levy flights and continuous random walk- all these are important for turbulent systems.