Pattern formation and control in complex networks

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Introduction

A brief introduction on pattern formation in continuous media and the networks.

The pioneer Alan Turing

THE CHEMICAL BASIS OF MORPHOGENESIS

By A. M. TURING, F.R.S. *University of Manchester*

(*Received* 9 *November* 1951—*Revised* 15 *March* 1952)

It is suggested that a system of chemical substances, called morphogens, reacting together and diffusing through a tissue, is adequate to account for the main phenomena of morphogenesis. Such a system, although it may originally be quite homogeneous, may later develop a pattern or structure due to an instability of the homogeneous equilibrium, which is triggered off by random disturbances. Such reaction-diffusion systems are considered in some detail in the case of an isolated ring of cells, a mathematically convenient, though biologically unusual system. The investigation is chiefly concerned with the onset of instability. It is found that there are six essentially different forms which this may take. In the most interesting form stationary waves appear on the ring. It is suggested that this might account, for instance, for the tentacle patterns on *Hydra* and for whorled leaves. A system of reactions and diffusion on a sphere is also considered. Such a system appears to account for gastrulation. Another reaction system in two dimensions gives rise to patterns reminiscent of dappling. It is also suggested that stationary waves in two dimensions could account for the phenomena of phyllotaxis.

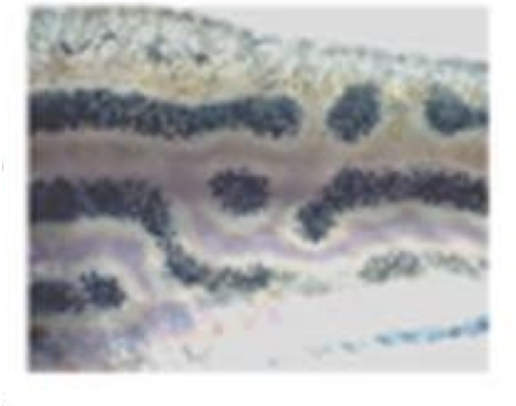
The purpose of this paper is to discuss a possible mechanism by which the genes of a zygote may determine the anatomical structure of the resulting organism. The theory does not make any new hypotheses; it merely suggests that certain well-known physical laws are sufficient to account for many of the facts. The full understanding of the paper requires a good knowledge of mathematics, some biology, and some elementary chemistry. Since readers cannot be expected to be experts in all of these subjects, a number of elementary facts are explained, which can be found in text-books, but whose omission would make the paper difficult reading.

$$\dot{u} = f(u,v) + \sigma_u \nabla^2 u\,,$$
$$\dot{v} = g(u,v) + \sigma_v \nabla^2 v\,.$$

Self-organization far from equilibrium

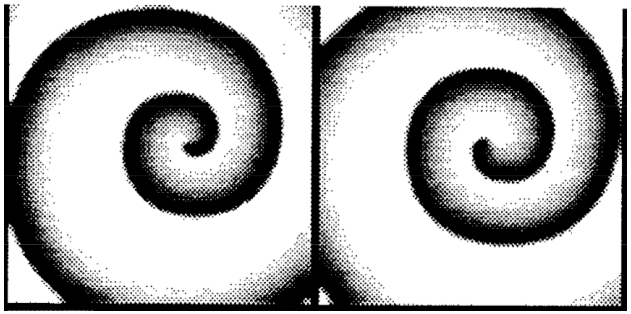
Reaction-diffusion systems support a wealth of self-organization phenomena

Turing patterns



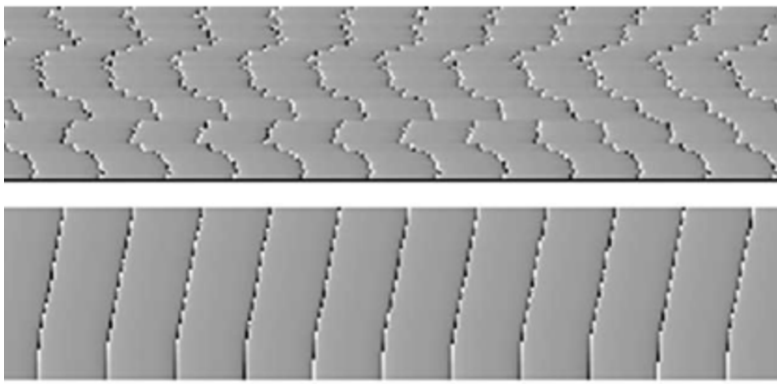
Nakamasu, Takahashi, Kanbe, Kondoab,PNAS 106 (2009)

Spiral waves



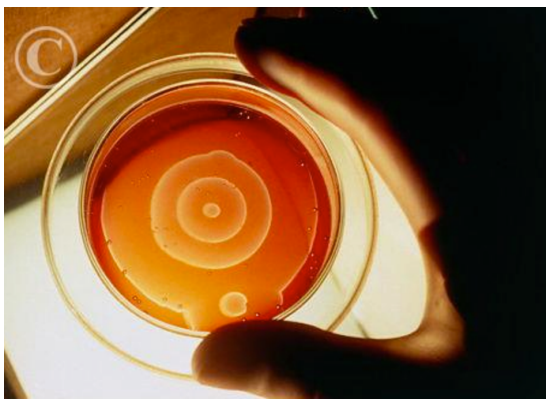
Bär, Gottschalk, Eiswirth, Ertl, J. Chem. Phys. (1993)

Synchronization



Osipov, Kurths, Zhou, *Synchronization in Oscillatory Networks*, Berlin 2007

Propagating fronts



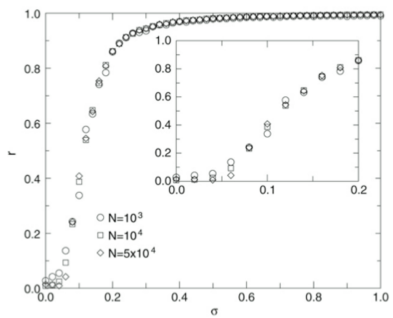
Wikipedia

Collective dynamics in complex networks

The nodes and the links of such networks may represent:

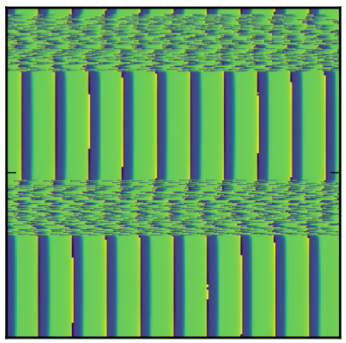
- Individual habitats and dispersal connections between them
- Electro-chemical units which are connected with resistors
- Neural cells and the synapses connecting them
- Individuals which interact via their social networks

Synchronization of coupled oscillators in networks:



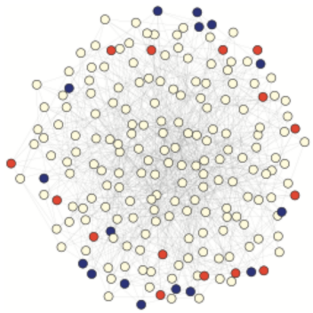
Arenas et al, *Synchronization in complex networks*, Phys Rep (2008)

Chimera states in non-local rings of coupled oscillators:



Kouvaris, Hizanidis et al, Chaos (2016)

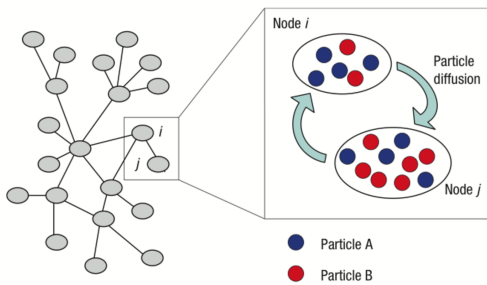
Turing patterns in network-organized activator-inhibitor systems:



Nakao, Mikhailov, Nat Phys (2010)

Epidemic spreading in networks:

(SIR: susceptible, infected, removed; SIS...)



Colizza, Pastor-Satorras, Vespignani, Nat Phys (2007)

Dynamics on networks

In order to study dynamics on networks we need

- A dynamical system (usually non-linear)
  - Monostable
  - Bistable
  - Oscillator
  - Ising model, Diffusion, Maps,...
- A network (usually complex)
  - Simplex
  - Multiplex/Multilayer
  - Modular
  - Colored
  - Metabolic networks, Facebook, Twitter, Mobility networks, The Brain, ...

$$\dot{\mathbf{u}}_i = \mathbf{F}(\mathbf{u}_i) + \sigma \sum_{j=1}^{N} H(\mathbf{u}_j)$$

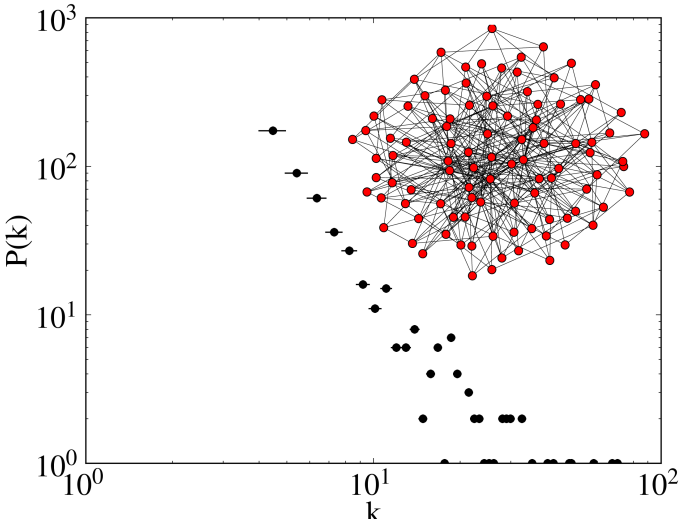
One can also study agent-based models in networks (Monte Carlo methods, etc...)

Complex networks

1. Scale-free

Power-law degree distribution

$P(k) \sim k^{-\gamma}$, typically $2 \leq \gamma \leq 3$



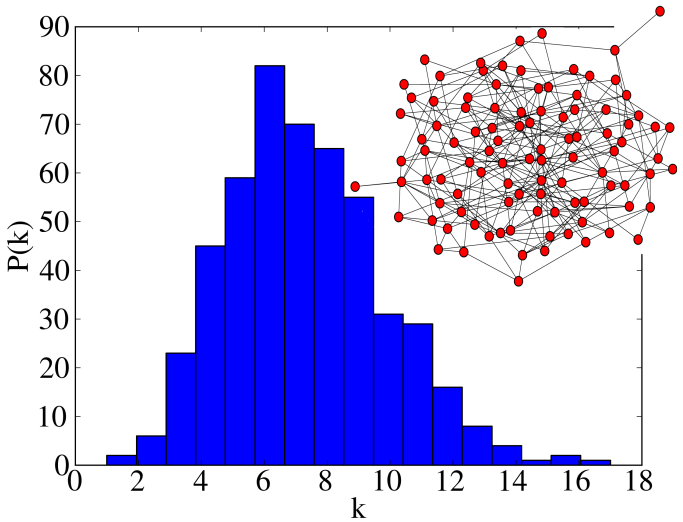
$\langle k \rangle = 2m$

(m is the number of links of a new added node)

2. Erdös-Rényi

Poisson degree distribution

$P(k) \sim \frac{(pN)^k e^{-pN}}{k!}$

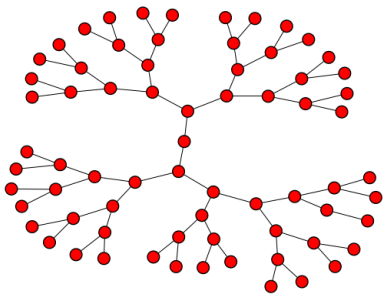


$\langle k \rangle = pN$

(p is the probability to have a link between 2 of N nodes)

3. k-ary trees

Each node has no ore than $k-1$ "children"



#nodes = #links-1

(the nodes population in each level grows exponentially with the distance from the root)

In Erdös-Rényi networks, the clustering coefficient ($\frac{\text{number of closed triplets}}{\text{number of all triplets}}$) is $C = \frac{\langle k \rangle}{N}$. If $N \to \infty$, then $C \to 0$, as required by the tree structure

Dorogovtsev, Mendes, *Evolution of Networks: From Biological Nets to the Internet and WWW*, Oxford 2003

Reaction-diffusion systems on networks

One-component reaction-diffusion systems:

$$\dot{u}_i = f(u_i) + D\sum_{j=1}^{N}(T_{ij}u_j - T_{ji}u_i)$$

- function $f(u_i)$ describes the local dynamics in the node i
- $\mathbf{T}$ is the adjacency matrix whose elements take the values $T_{ij} = 1$ if the nodes i and j are connected and $T_{ij} = 0$ otherwise.

In undirected networks the adjacency matrix is summetric $T_{ij} = T_{ji}$
In weighted networks T_{ij} can take values different than 0 and 1.

The RD system can also be written as

$$\dot{u}_i = f(u_i) + D\sum_{j=1}^{N}L_{ij}u_j$$

- $\mathbf{L}$ is the Laplacian matrix whose elements are given by $L_{ij} = T_{ij} - k_i\delta_{ij}$
- k_i is the number of connections (degree) of a node i $k_i = \sum_{j=1}^{N} T_{ji}$
- D is the rate of diffusive transport of u across the links.

For the 1D chains this equation reads as

$$\dot{u}_i = f(u_i) + D(u_{i-1} - 2u_i + u_{i+1})$$

Bistable networks

We discuss the formation mechanism of localized patterns in networks of coupled bistable elements.

The Schlögl bistable model in networks

Friedrich Schlögl (1917-2011) had proposed a hypothetical trimolecular chemical reaction with an autocatalytic state given by the stochiometric scheme:

$$A + 2U \underset{c_2}{\overset{c_1}{\rightleftharpoons}} 3U$$

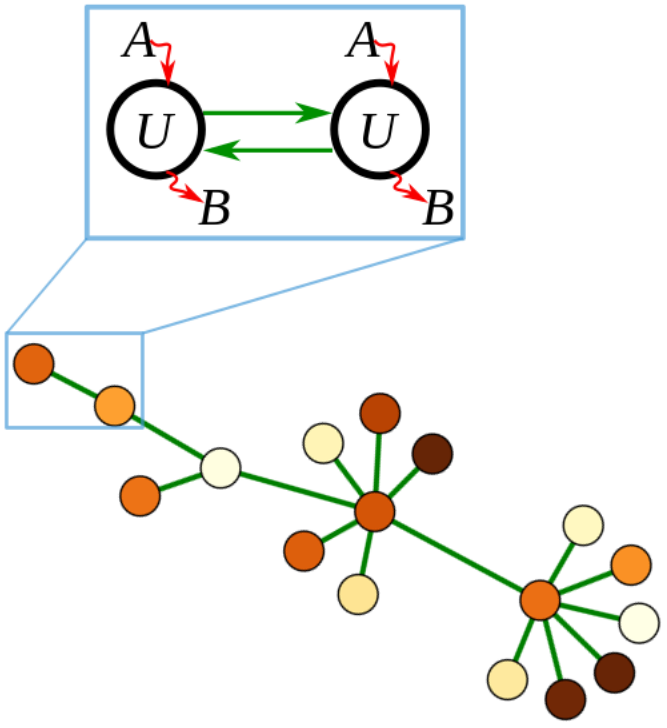
$$U \underset{c_4}{\overset{c_3}{\rightleftharpoons}} B$$

If the concentrations of the species A, B are constant, by influx of A into the system and outflux of B out of the system, the system is maintained in a nonequilibrium state.

The evolution of the species U concentration in this nonequilibrium state is given by,

$$\dot{u}(t) = f(u) = -(u - r_1)(u - r_2)(u - r_3)$$

F. Schlögl, Z. Physik 253, 147 (1972)



We address the questions:

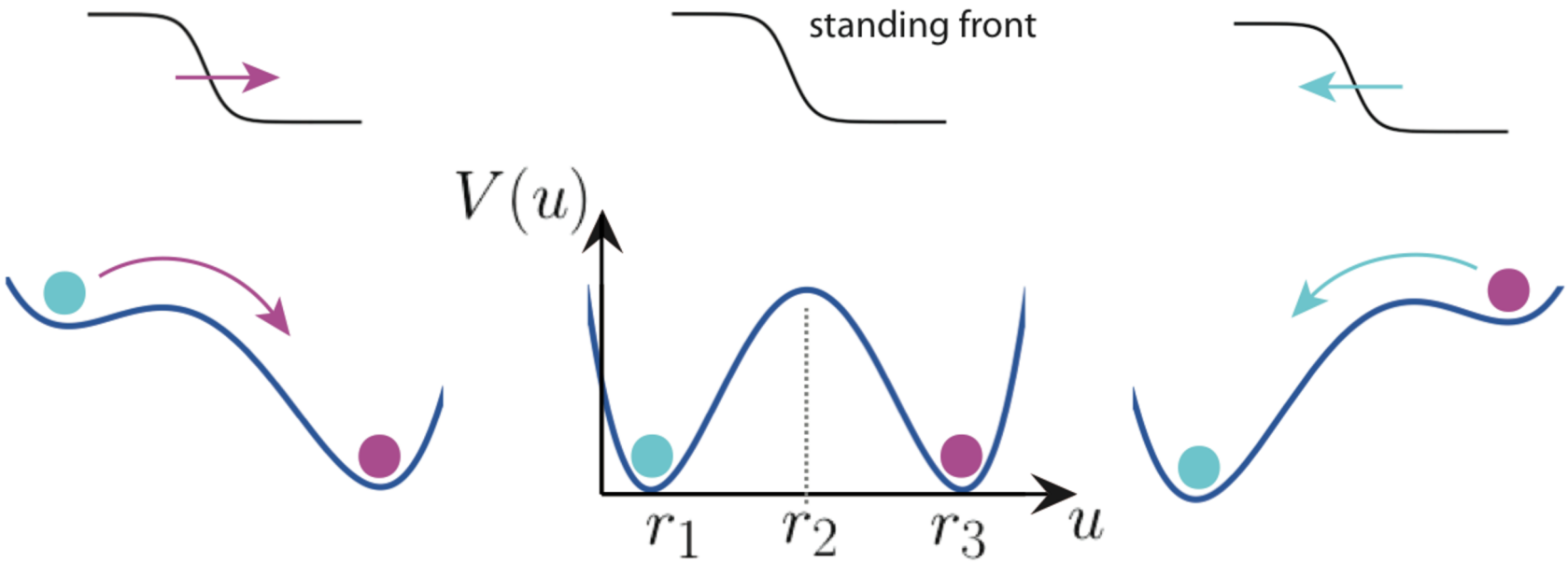
- How the topology of the network shapes the hosted dynamics?
- Can we **control** the dynamics and design new patterns?

Bistable dynamics / Bistable media

Local dynamics: $\frac{\partial}{\partial t}u(t) = f(u) = -(u - r_1)(u - r_2)(u - r_3)$

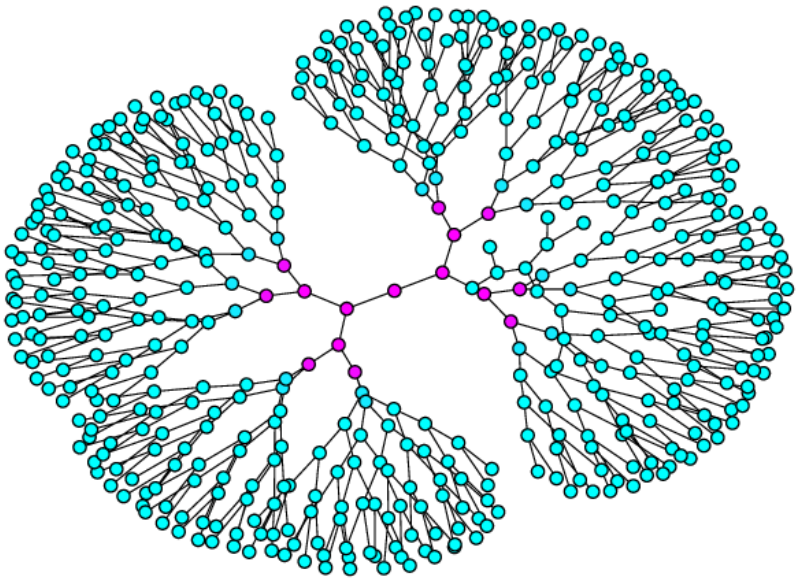
Chemical potential: $V(u) = -\int f(u)du$

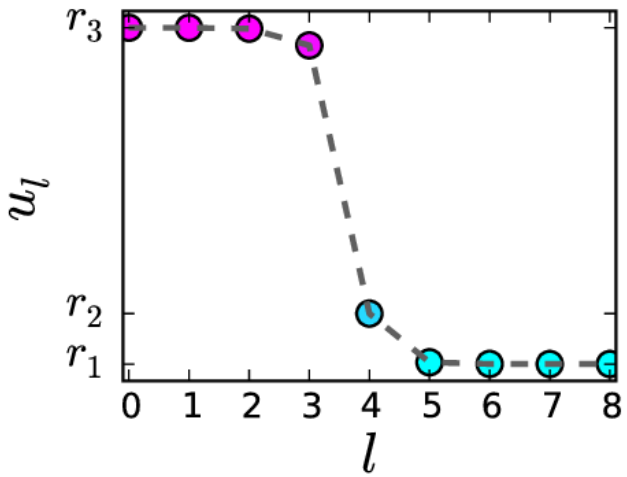
Bistable medium: $\frac{\partial}{\partial t}u(\mathbf{x}, t) = f(u) + D\nabla^2 u(\mathbf{x}, t)$



Regular trees with fixed branching factor

All nodes lying in the same distance from the root can be grouped into a single shell:





A node at the shell l should be diffusively coupled to $k-1$ nodes in the next shell $l+1$ and to just one node in the previous shell $l-1$.

The evolution of the activation level u_l is then described by:

$$\dot{u}_l = f(u_l) + D(u_{l-1} - u_l) + D(k-1)(u_{l+1} - u_l)$$

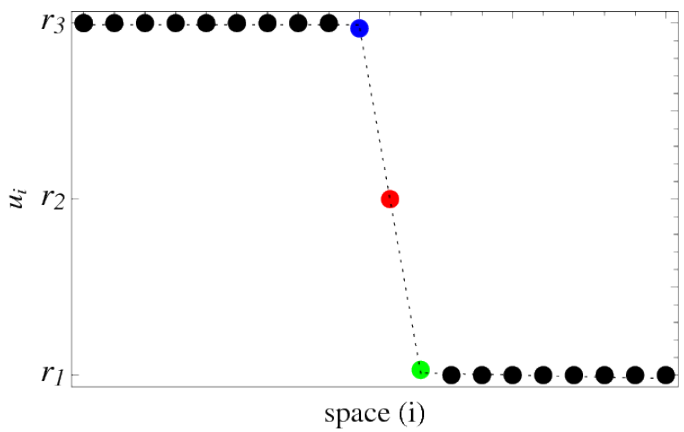
Discrete 1D chains

$$\dot{u}_l = f(u_l) + D\left[u_{l-1} - 2u_l + u_{l+1}\right]$$

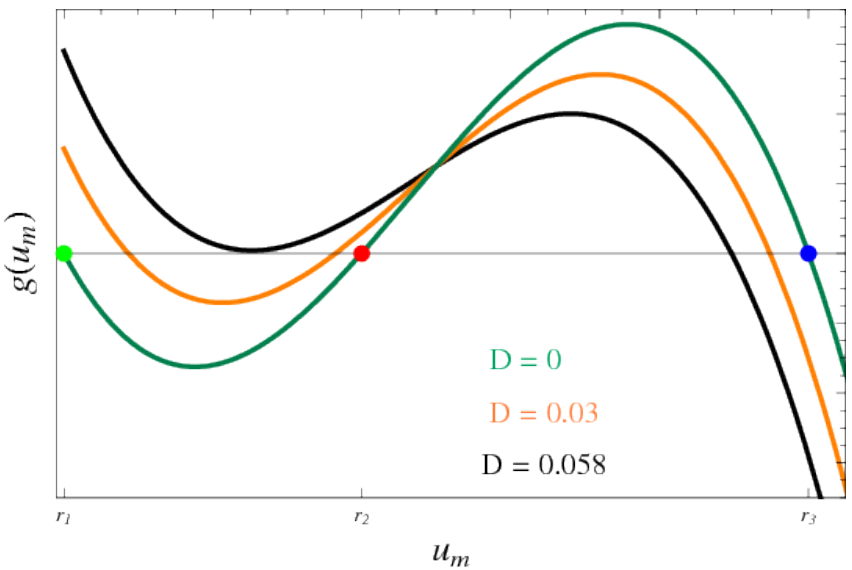
Mitkov, Kladko, Pearson, PRL 81 (1998)

A stationary front (pinned) solution is found for $\dot{u}_l = 0$.

$u_{m-1} \approx r_3$ and $u_{m+1} \approx r_1$



$$g(u_m) \equiv \dot{u}_m = f(u_m) + D(r_3 - 2u_m + r_1)$$

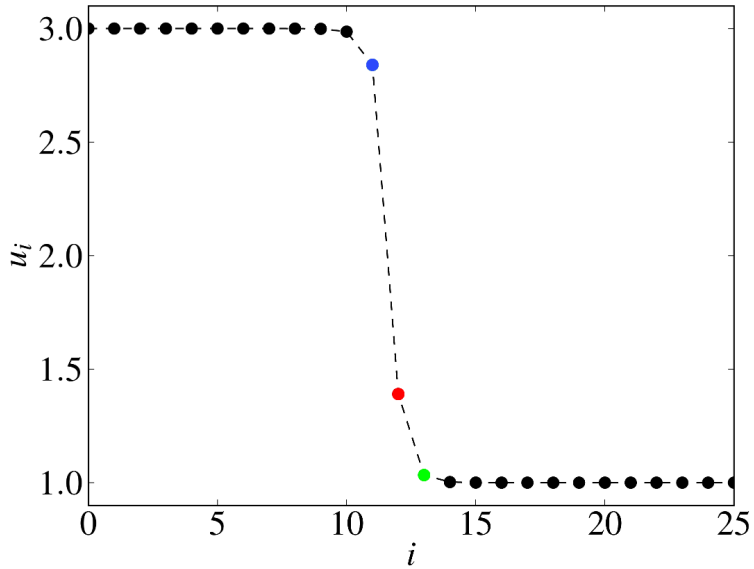


Regular trees

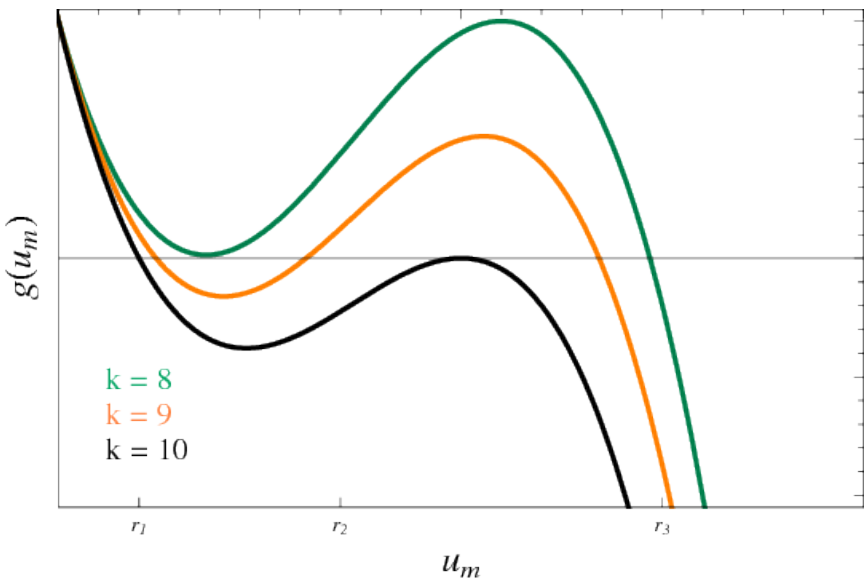
$$\dot{u}_l = f(u_l) + D\left[u_{l-1} - k u_l + (k-1) u_{l+1}\right]$$

k is the nodes degree and thus takes only integer values. However, in our theoretical approximation k is treated as a continuous parameter.

actual pinned front from the simulations



$$g(u_m) \equiv \dot{u}_m = f(u_m) + D\left[r_3 - k u_m + (k-1) r_1\right]$$



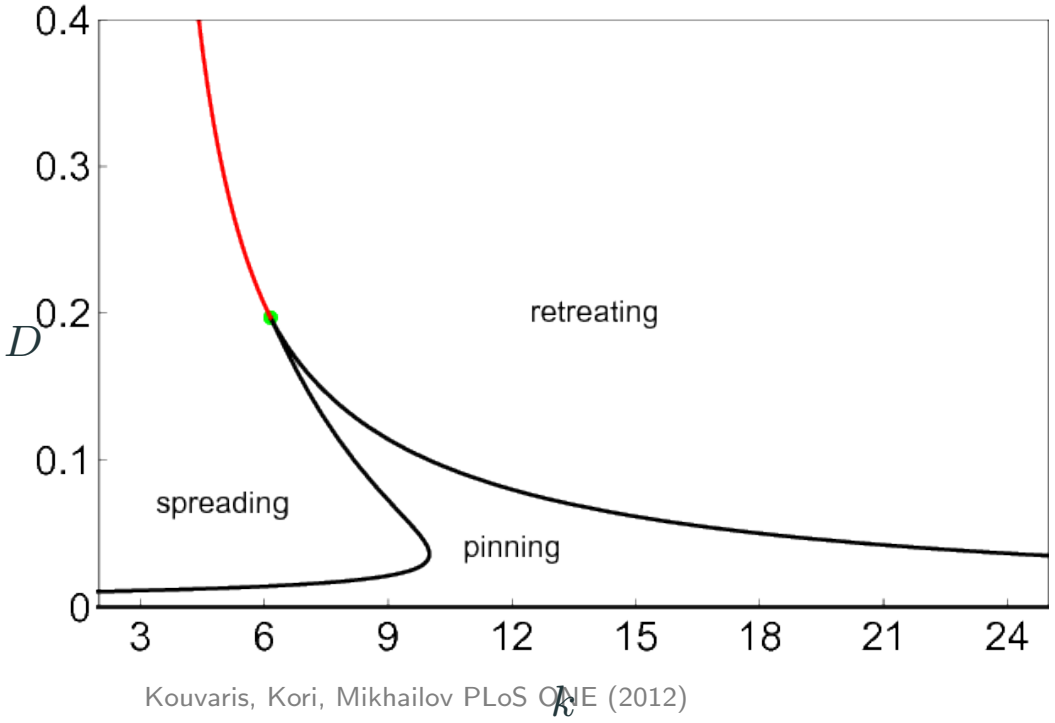
In the same manner we assume that

$u_{m-1} \approx r_3$ and $u_{m+1} \approx r_1$

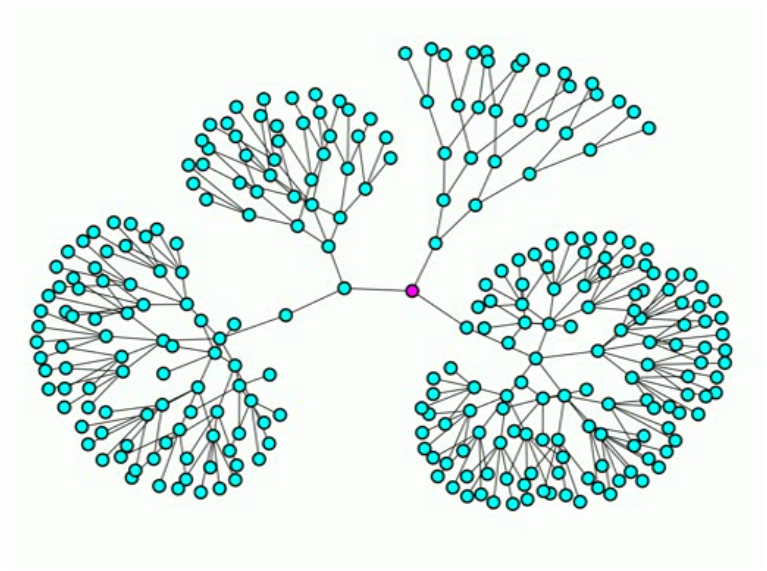
Kouvaris, Kori, Mikhailov PLoS ONE (2012)

Bifurcation diagram

$(r_2 = 1.2)$

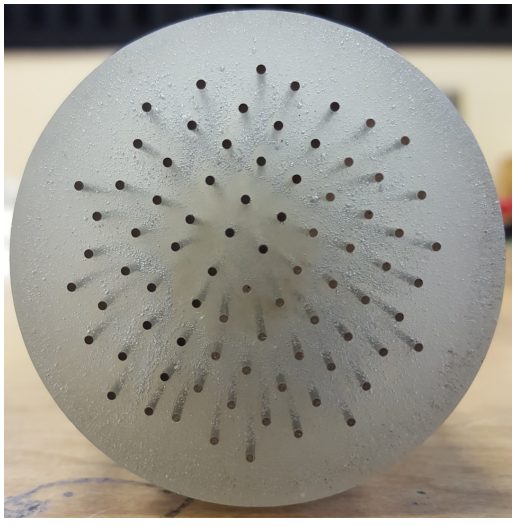


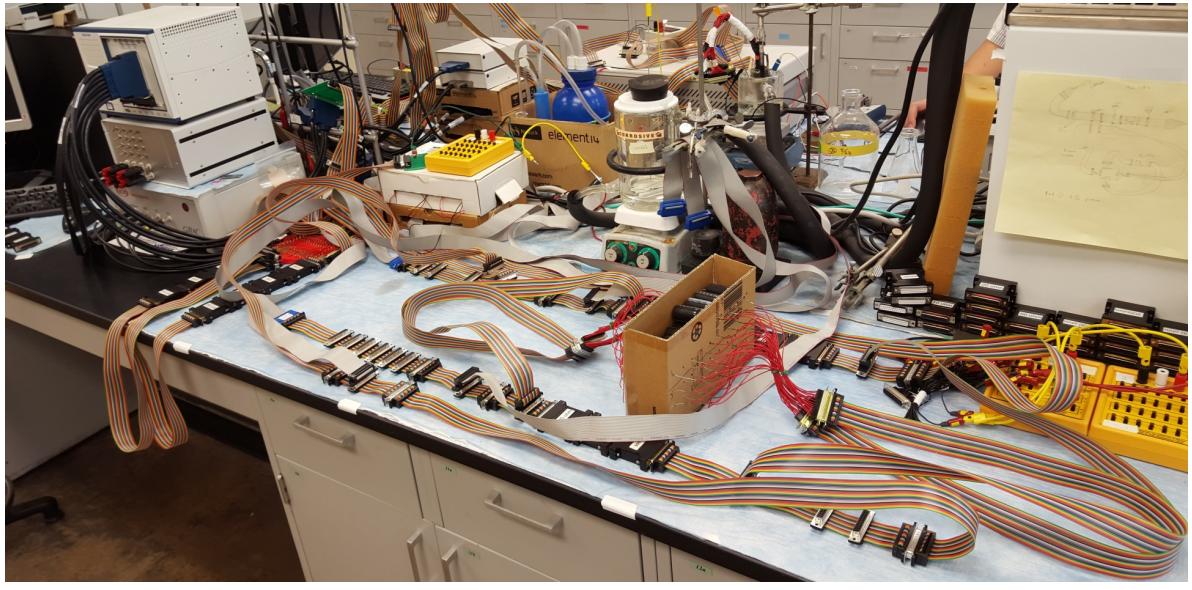
Kouvaris, Kori, Mikhailov PLoS ONE (2012)

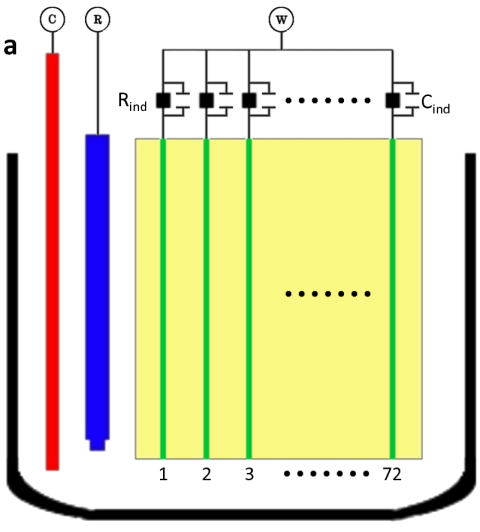


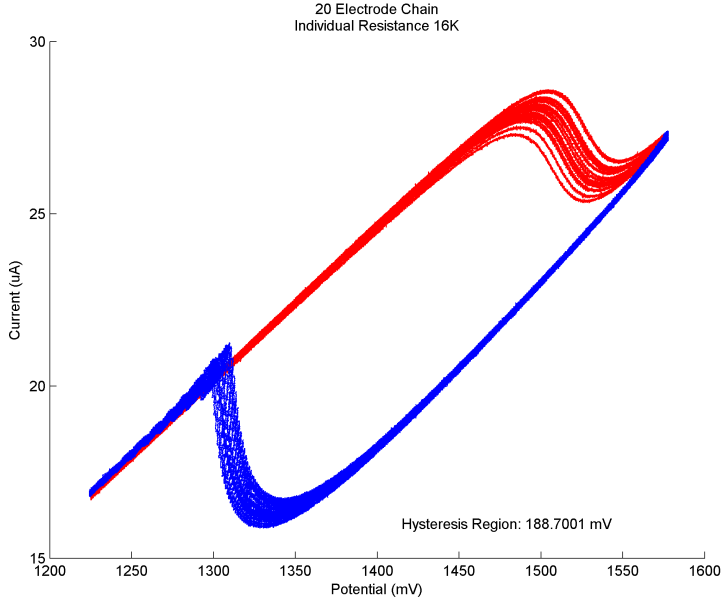
$(D = 0.05)$

Electrochemical experiments



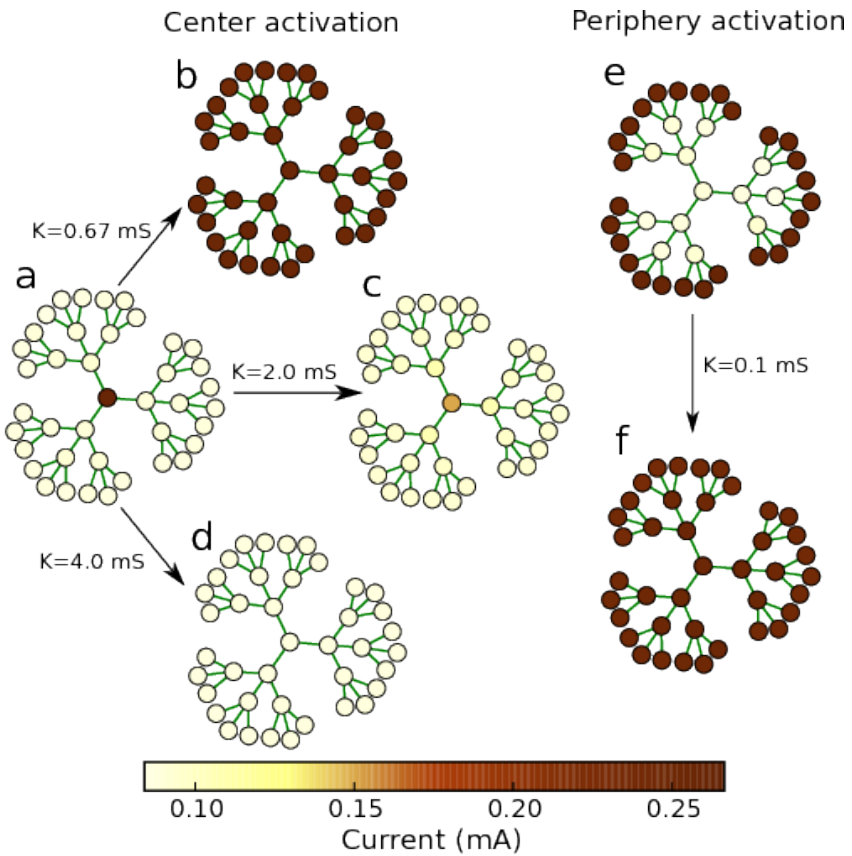


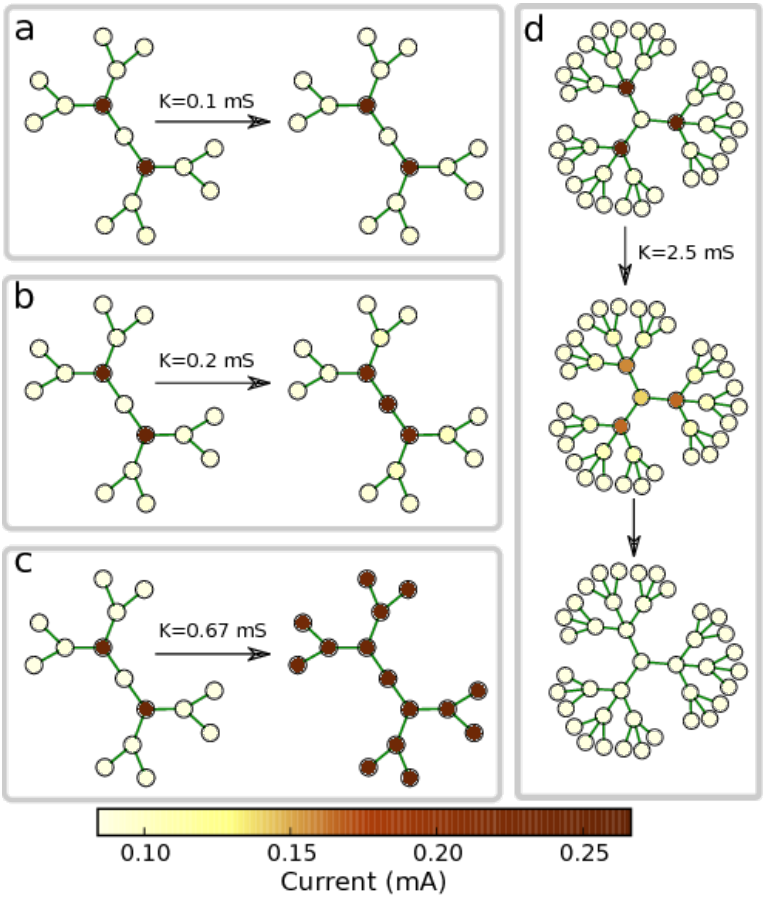




Kouvaris, Sebek, Mikhailov, Kiss, Angewandte Chemie **128** (2016)

Experimental observation of patterns in bistable trees





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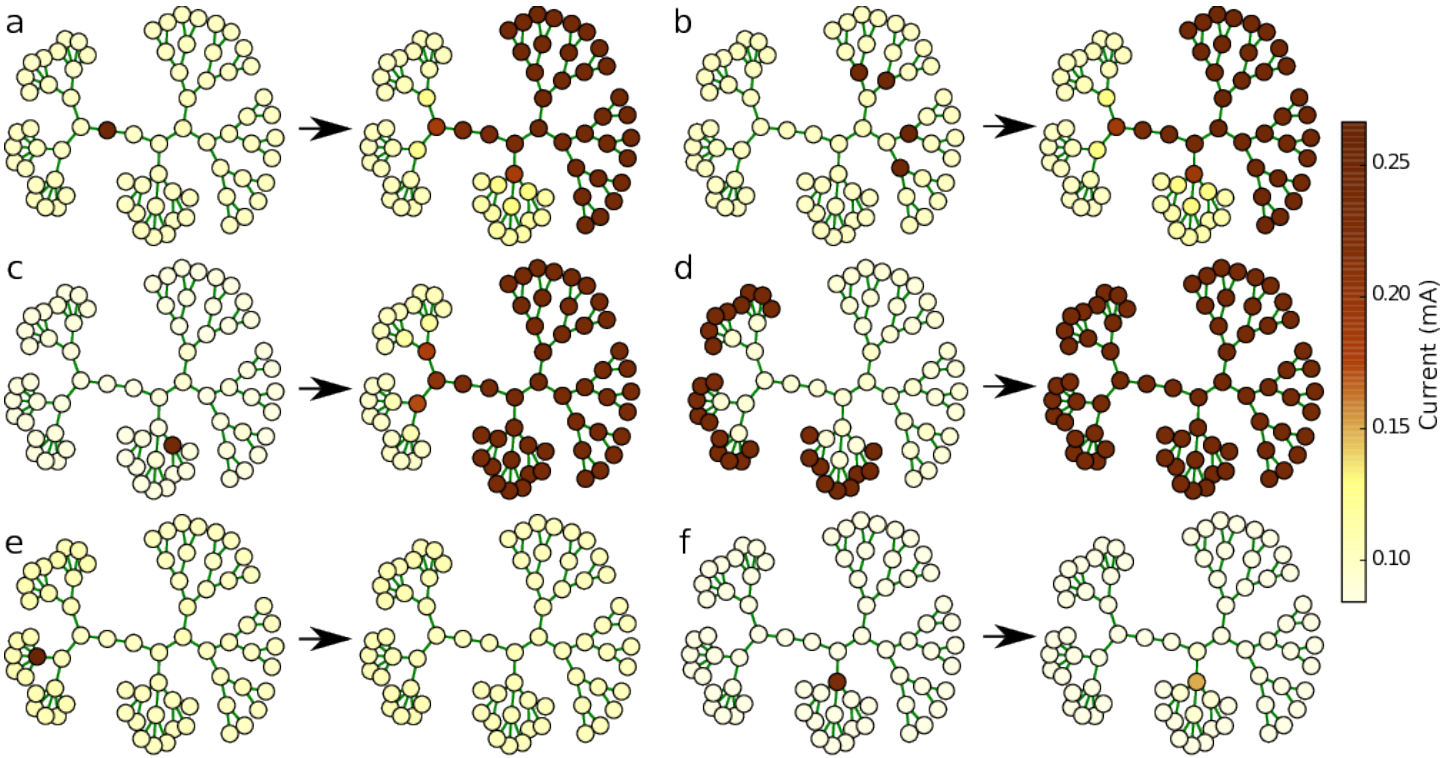


Kouvaris, Sebek, Mikhailov, Kiss, Angewandte Chemie **128** (2016)

Control of bistable networks

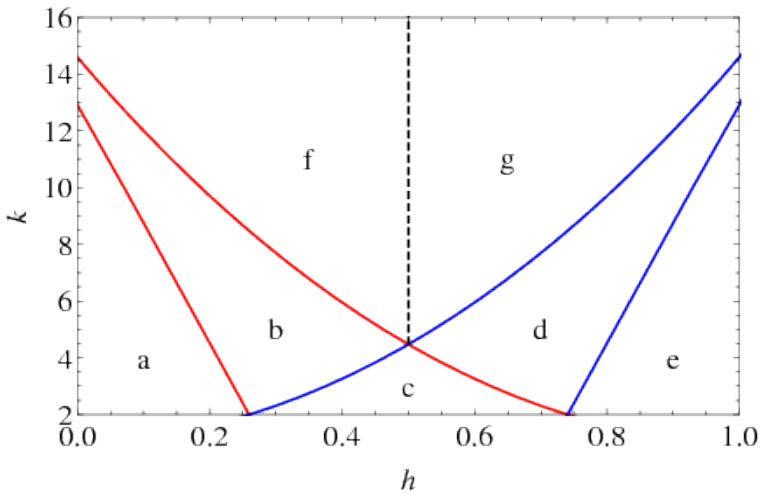
We discuss control schemes and the purposeful design of stationary and oscillatory patterns in networks of bistable nodes.

Control of pattern formation in bistable trees

Let us now consider the local bistable dynamics

$$f(u,h) = u(h-u)(u-1)\,,$$

where h specifies the activation threshold.



Global negative feedback

The parameter

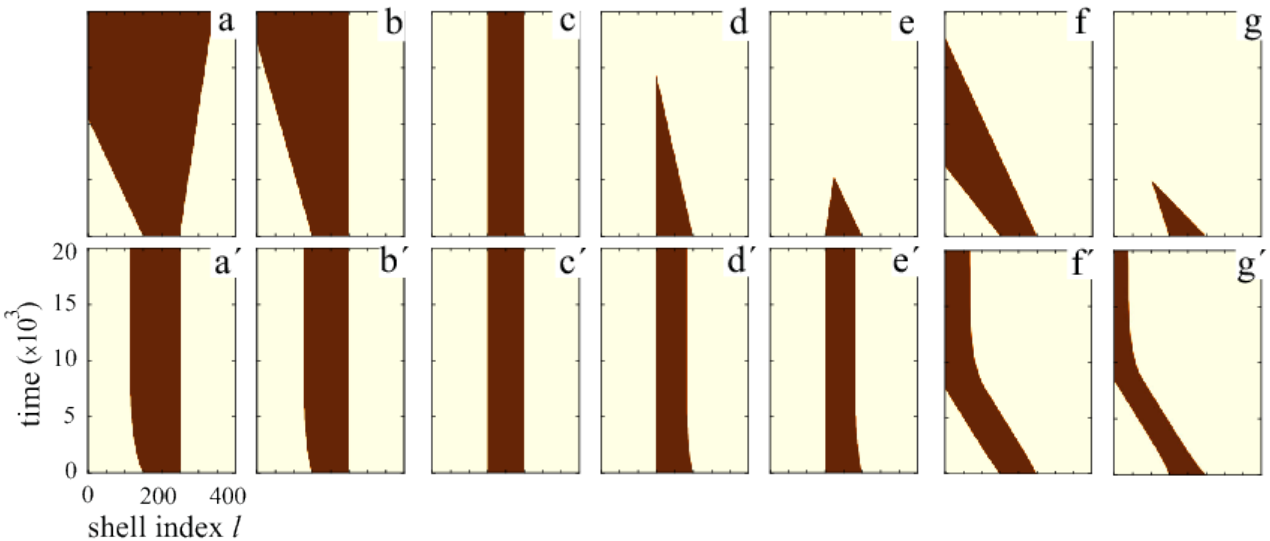
$$h = h_0 + \mu(S(t) - S_0)\,,$$

depends on the total activation

$$S(t) = \sum_{j=1}^{N} u_j(t),$$

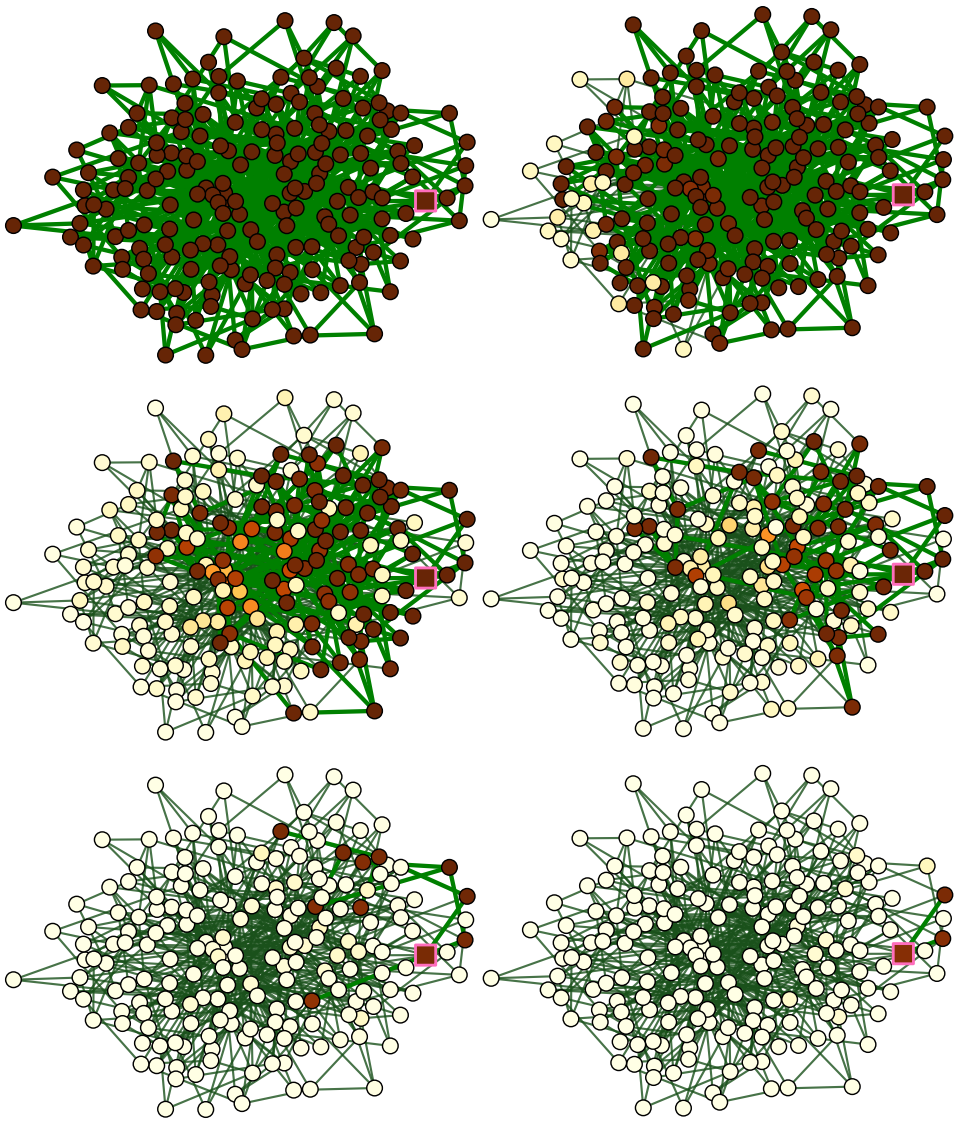
and increases when more nodes are activated, so that a negative feedback is realized.

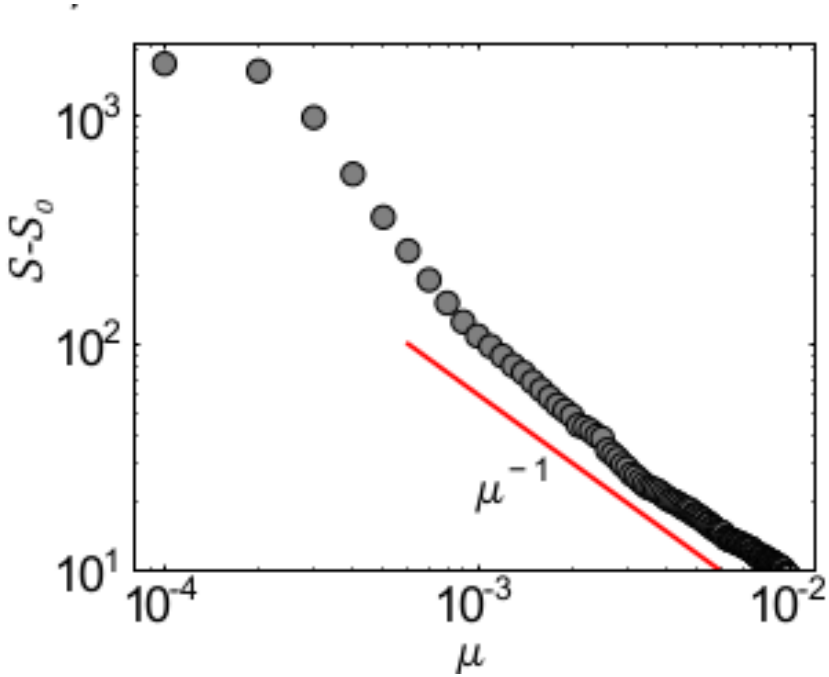
Krisher, Mikhailov, PRL (1994)



Kouvaris, Mikhailov EPL (2013)

Global control in random networks





Kouvaris, Mikhailov EPL (2013)

Global control in random networks

The size S of a stationary pattern grows like,

$$\dot{S} = c(h)\,,$$

where $c(h)$ is the propagation velocity of a front. When a stationary pattern is established, its size should not change with time and thus $\dot{S} = 0$, namely

$$\begin{aligned} c(h) &= 0 \\ c\left[h_0 + \mu(S - S_0)\right] &= 0 \end{aligned}$$

Let us assume here that the velocity is dependent on k as $c(h) = D(c_0 - \sqrt{k}h)$. Then the intensity of the feedback can be given by

$$\mu = \frac{c_0 - \sqrt{k}h_0}{\sqrt{k}(S - S_0)}\,.$$

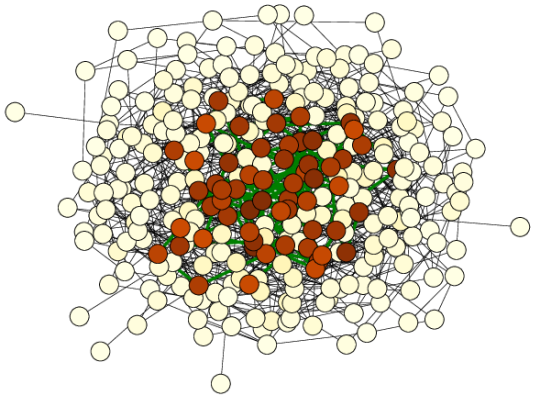
where we see that $\mu \propto 1/(S - S_0)$. Then by taking into account that $\langle u \rangle \propto (S - S_0)$ we conclude on the relation

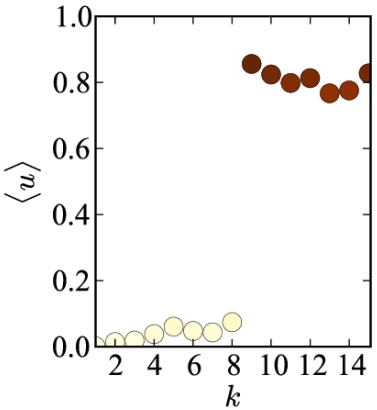
$$\langle u \rangle \propto \mu^{-1}$$

Local control in random networks

$h_i = h_0 + \mu H(k_i)(S - S_0)$ where $H(k_i)$ is the step function.

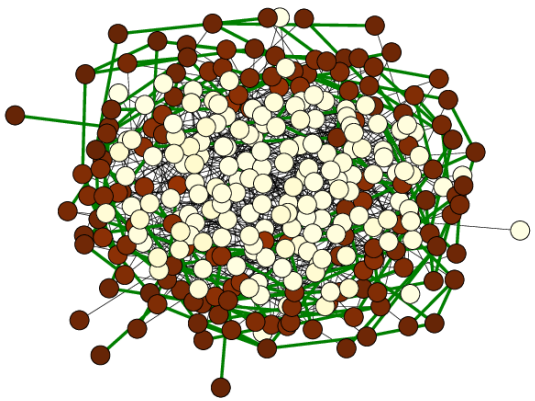
$H(k_i) = 1$ if $k < k_0$ and
$H(k_i) = 0$ if $k \geq k_0$

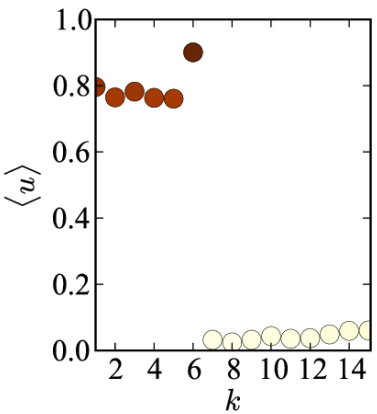




or

$H(k_i) = 1$ if $k > k_0$ and
$H(k_i) = 0$if $k \leq k_0$





Kouvaris, Mikhailov EPL (2013)

Excitable networks

Networks of coupled excitable nodes (e.g. neurons) give rise to a rich spationtemporal behaviour. Here we study propagation failure of excitations as a result of the nodes connectivity degree.

Fitzhugh-Nagumo dynamics on the trees

Let us now assume the local excitable dynamics given by:

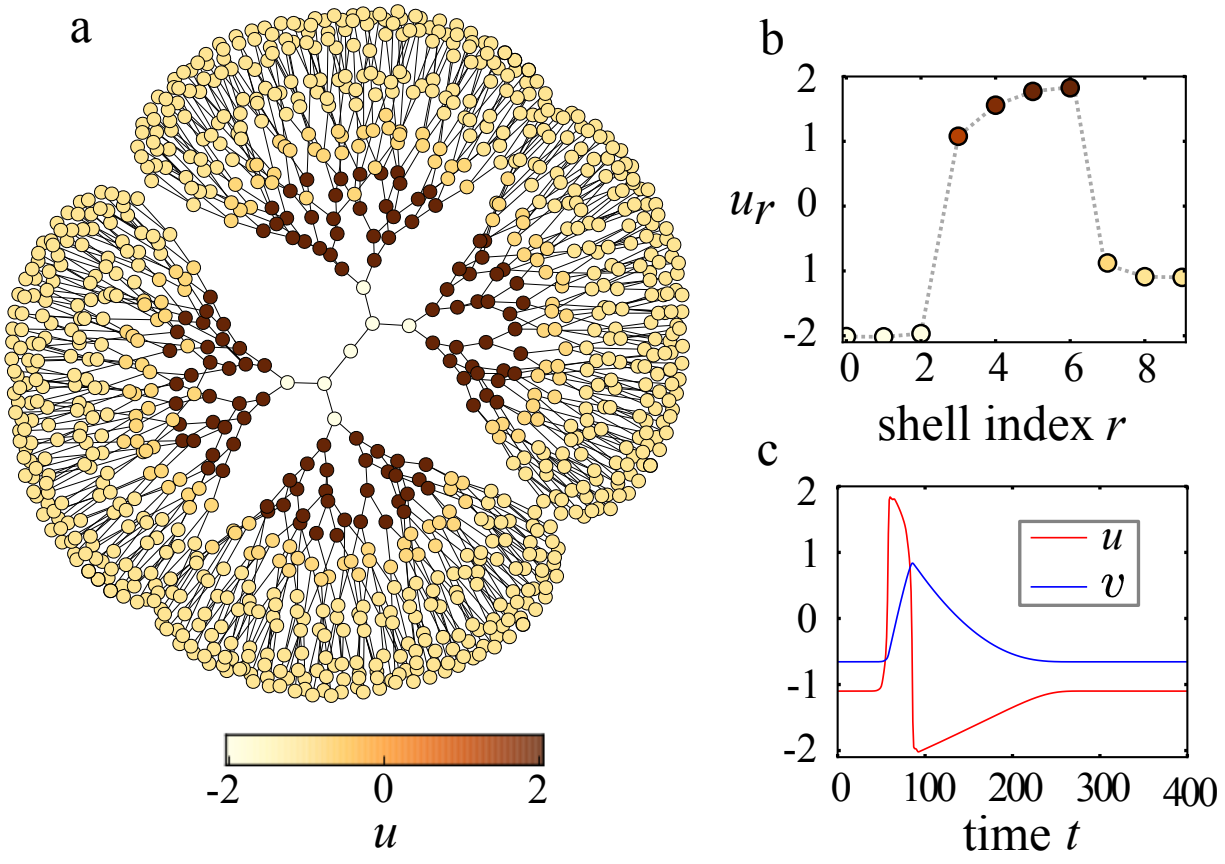
$$
\begin{aligned}
f(u,v) &= u - \frac{u^3}{3} - v \\
g(u,v) &= u - \beta
\end{aligned}
$$

and the network-organized activator-inhibitor system:

$$
\begin{aligned}
\dot{u}_i &= f(u_i, v_i) + D \sum_{j=1}^{N} L_{ij} u_j \,, \\
\dot{v}_i &= \varepsilon g(u_i, v_i) \,.
\end{aligned}
$$

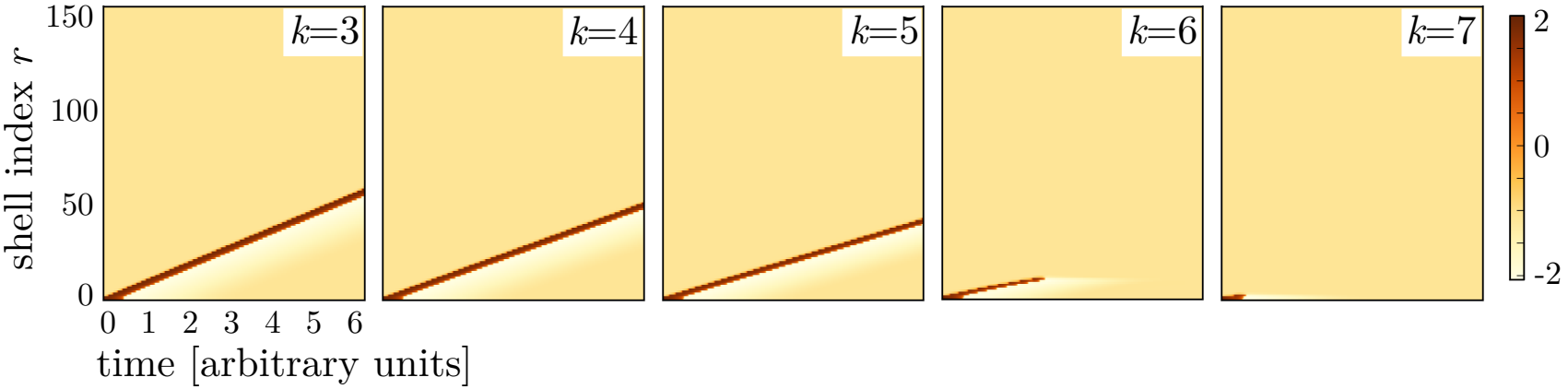
On a regular tree this system reads,

$$
\begin{aligned}
\dot{u}_r &= f(u_r, v_r) + D[u_{r-1} - k u_r + (k-1) u_{r+1}] \,, \\
\dot{v}_r &= \varepsilon g(u_r, v_r) \,.
\end{aligned}
$$

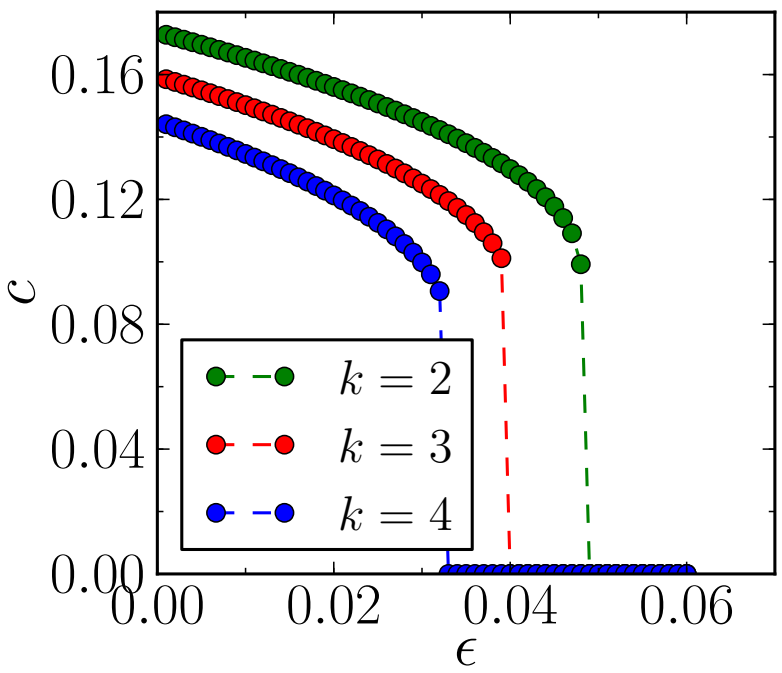


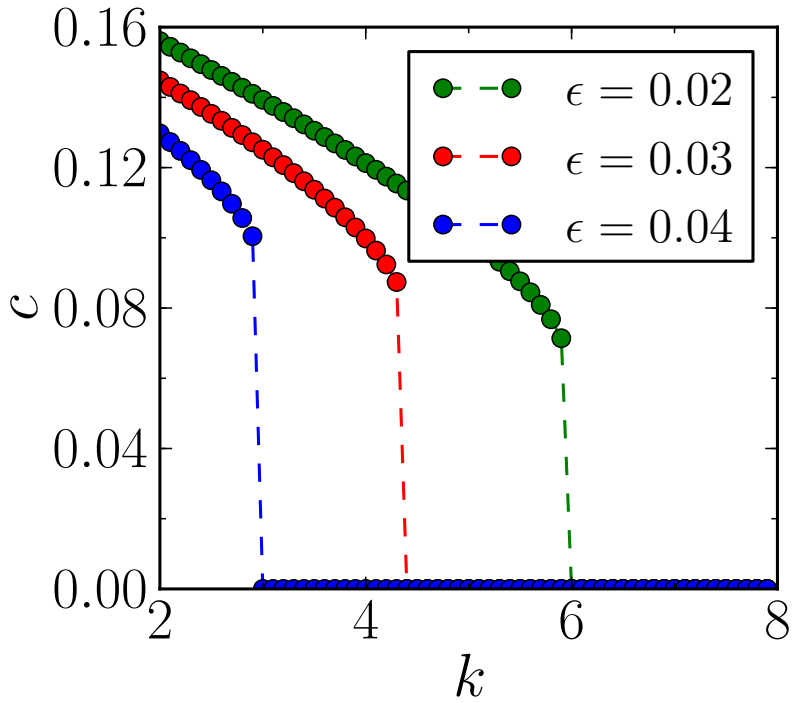
Kouvaris, Isele, Mikhailov, Schöll EPL (2014)

Propagation failure of excitation waves



$\varepsilon = 0.02,\ D = 0.04,\ \beta = -1.1$ movie ($k = 6$)

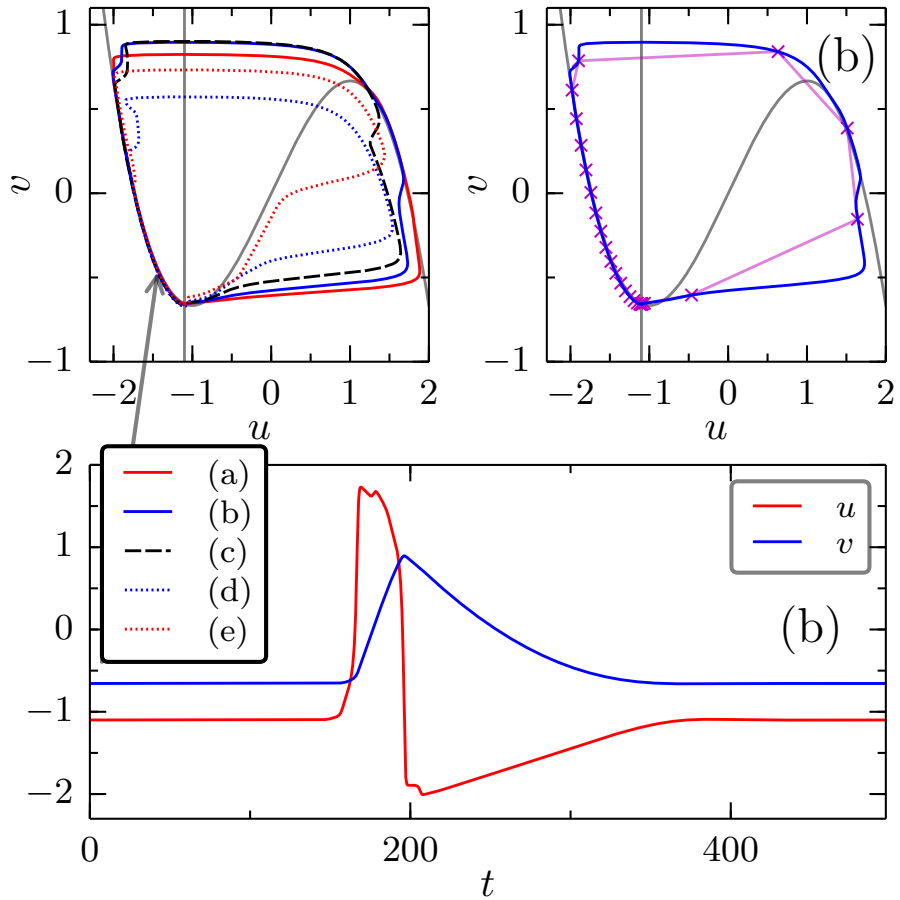


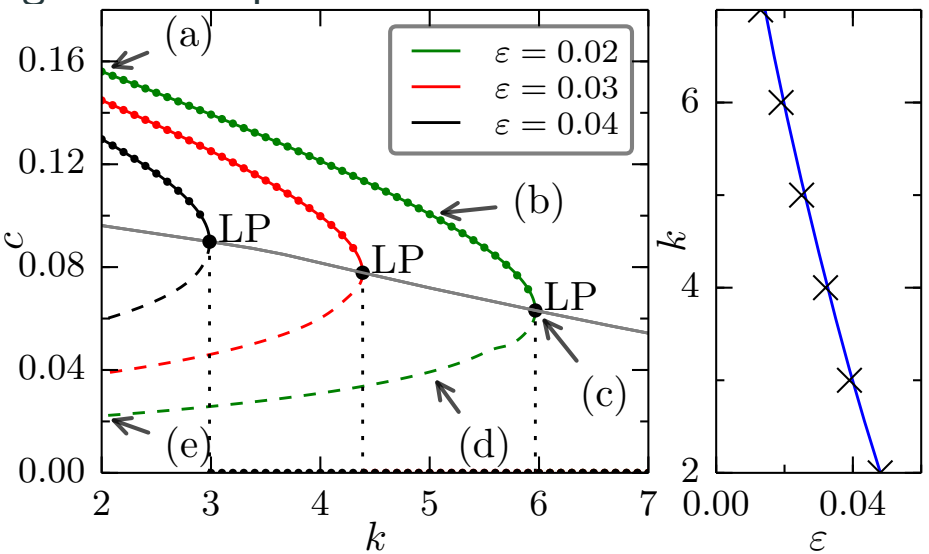


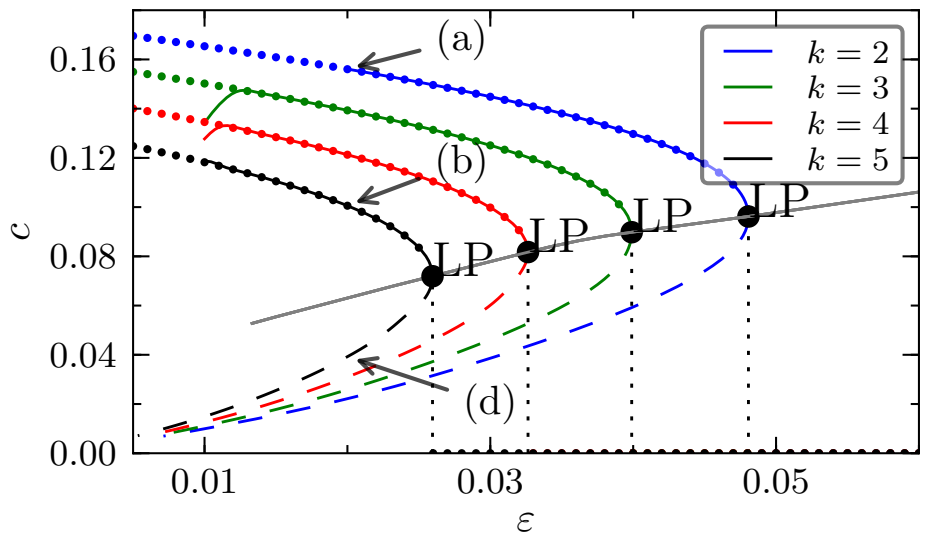
Kouvaris, Isele, Mikhailov, Schöll EPL (2014)

Saddle-node bifurcations

An excitation wave on this ring of N shells is a periodic orbit of $2N$ ODEs, of the system and thus can easily be continued, using AUTO-07p.







movie (b) movie (LP)

Kouvaris, Isele, Mikhailov, Schöll EPL (2014)

Kinematical theory of excitation waves in the trees

$$\dot{u}_r = f(u_r, v_r) + D[u_{r-1} - ku_r + (k-1)u_{r+1}], \quad \dot{v}_r = \varepsilon g(u_r, v_r)$$

we substitute u_{r-1} and u_{r+1} with their Taylor expansions:

$$\begin{aligned} u_{r-1} &\approx u_r - \nabla u + \frac{1}{2}\Delta u \\ u_{r+1} &\approx u_r + \nabla u + \frac{1}{2}\Delta u\,. \end{aligned}$$

The continuous system then reads,

$$\begin{aligned} \dot{u}(\mathbf{x}) &= f(u,v) + \frac{Dk}{2}\Delta u + D(k-2)\nabla u\,, \\ \dot{v}(\mathbf{x}) &= \varepsilon g(u,v)\,. \end{aligned}$$

By introducing the moving reference frame

$$\xi = r - ct$$

where the profile of the wave is stationary we can write,

$$\begin{aligned} -[c + D(k-2)]u' &= f(u,v) + \frac{kD}{2}u''\,, \\ -cv' &= \varepsilon g(u,v)\,, \end{aligned}$$

where $u = u(\xi)$ and $v = v(\xi)$. Subsequently, if we replace ε by ε^* in the latter equation, where,

$$\varepsilon^* = \varepsilon\left[1 + \frac{D(k-2)}{c}\right],$$

we take the system of equations,

$$\begin{aligned} -[c + D(k-2)]u' &= f(u,v) + \frac{kD}{2}u''\,, \\ -[c + D(k-2)]v' &= \varepsilon^* g(u,v)\,. \end{aligned}$$

Kouvaris, Isele, Mikhailov, Schöll EPL (2014) d

Saddle-node bifurcations

Therefore, in the trees with ε^*, the propagation velocity $c(\varepsilon^*)$ of an excitation wave will be equal to the sum $c + D(k-2)$, i.e.,

$$c(\varepsilon^*) = c + D(k-2)\,.$$

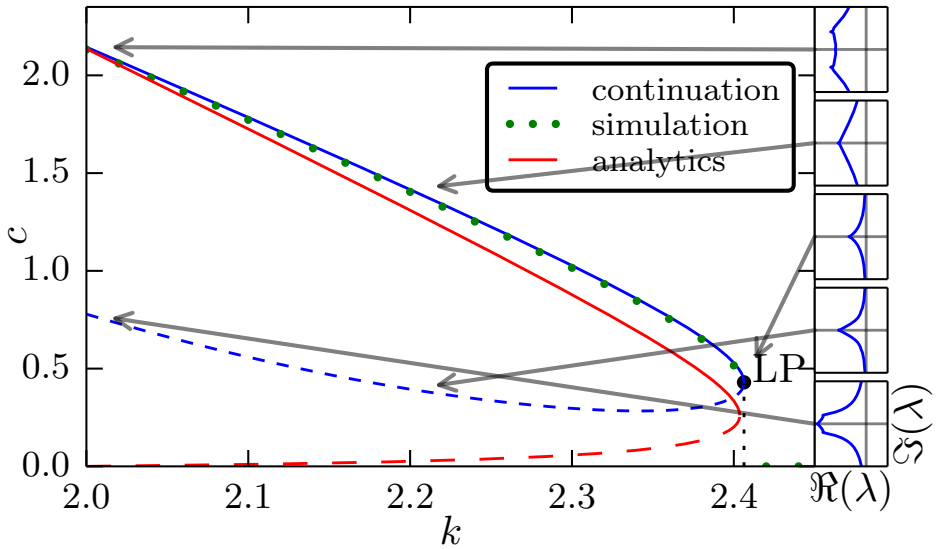
Numerical simulations have revealed that for very small ε the velocity c depends linearly on this parameter, i.e.,

$$c(\varepsilon) = c_0(1 - \chi\varepsilon)\,,$$

where χ is a numerical factor independent of ε. Substituting expressions for ε^* and $c(\varepsilon^*)$ into the latter equation we find an approximate analytical expression for the velocity of an excitation wave,

Continuation of the profile equations,

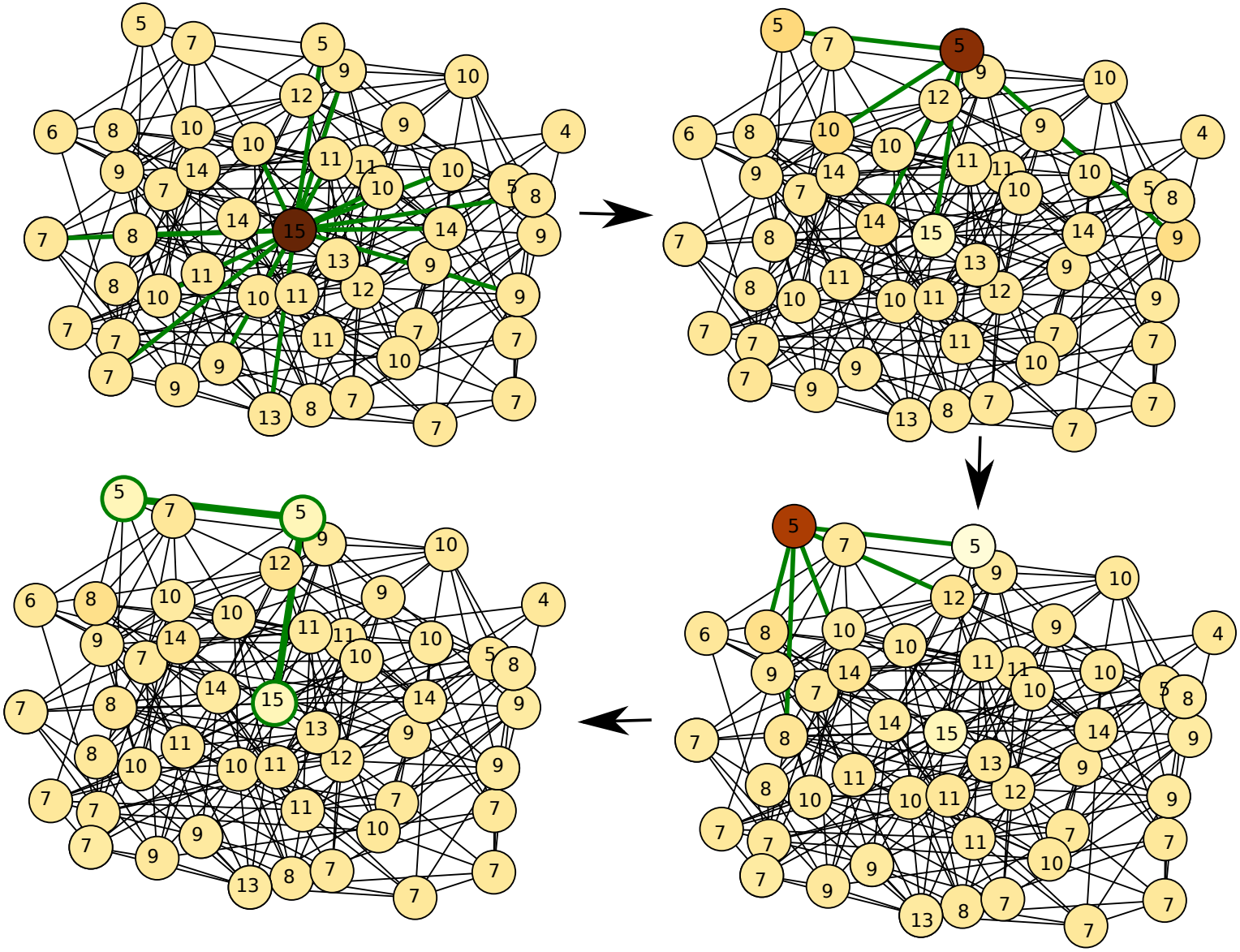
$$\begin{aligned} u' &= w \\ v' &= -c^{-1} g(u,v) \\ w' &= -\frac{2}{Dk}\left\{f(u,v) + [c + D(k-2)]\, w\right\}. \end{aligned}$$



$$\begin{aligned} c &= \frac{1}{2}\left[c_0(1-\varepsilon\chi) - D\,(k-2)\right] \\ &\quad \pm \frac{1}{2}\left\{\left[D(k-2) - c_0\,(1-\varepsilon\chi)\right]^2 - 4c_0 D\varepsilon\chi\,(k-2)\right\}^{1/2}. \end{aligned}$$

Kouvaris, Isele, Mikhailov, Schöll EPL (2014)

Random networks



$\varepsilon = 0.02,\ D = 0.04,\ \beta = -1.1$

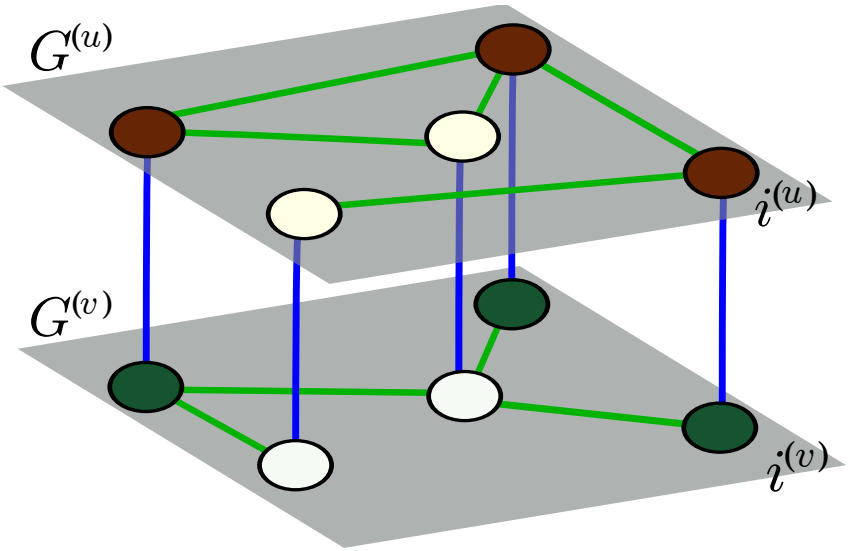
Kouvaris, Isele, Mikhailov, Schöll EPL (2014)

Ecological multiplex

Ecological networks may have different type of links/pathways, which are used by different species. This allows for their interpretation in the framework of multiplex networks, where Turing patterns can develop.

Prey-predator systems with different migration pathways

Activator-inhibitor dynamics on multiplex networks



$$\dot{u}_i = f(u_i, v_i) + \sigma^{(u)} \sum_{j=1}^{N} L_{ij}^{(u)} u_j \,,$$

$$\dot{v}_i = g(u_i, v_i) + \sigma^{(v)} \sum_{j=1}^{N} L_{ij}^{(v)} v_j \,.$$

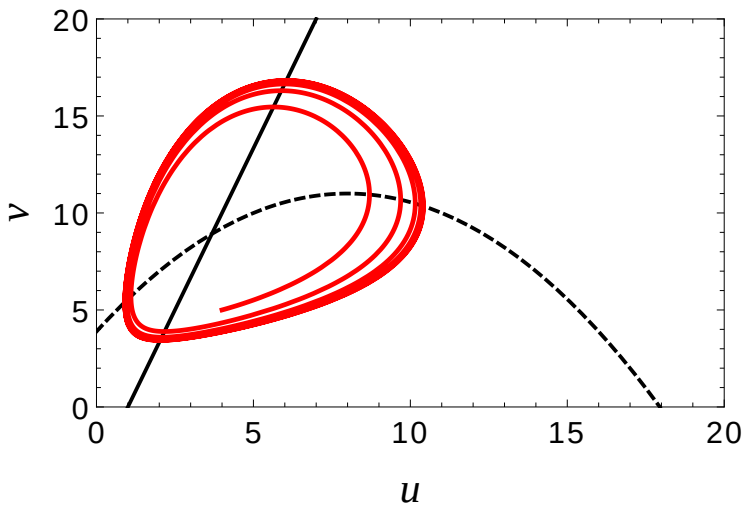
- u_i, v_i are the densities of activator and inhibitor in the nodes $i^{(u)}$ and $i^{(v)}$
- Functions f, g define the dynamics of activator and inhibitor in the nodes $i^{(u)}$ and $i^{(v)}$
- $L^{(u)}$ and $L^{(v)}$ are the Laplacian matrices of the networks $G^{(u)}$ and $G^{(v)}$
- $\sigma^{(u)}$ and $\sigma^{(v)}$ are the rates of diffusional mobility

This general formulation of the Multiplex-RD systems can be used in any different context of activator-inhibitor dynamics, e.g. neurodynamics, prey-predator, chemical reactions, etc., where different species move within different networks.

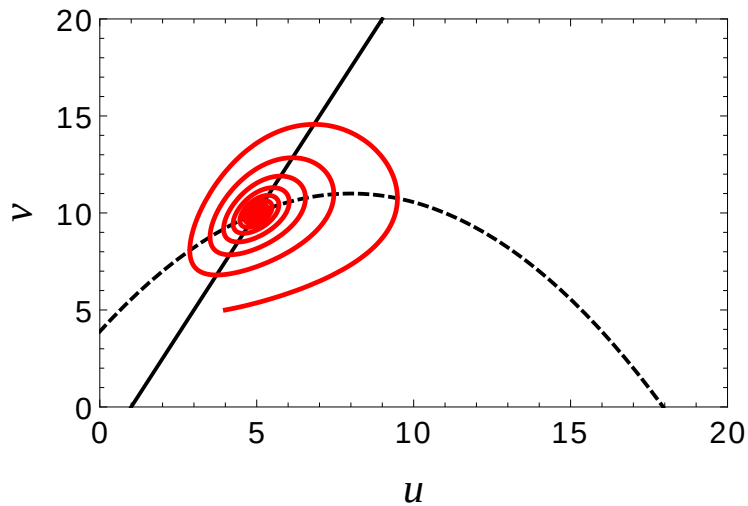
The Mimura-Murray ecological model

We consider the Mimura-Murray ecological model
(activator $u \to$ prey and inhibitor $v \to$ predator):

$$\begin{aligned} f(u,v) &= \left(\frac{a+bu-u^2}{c} - v\right) u\,, \\ g(u,v) &= (u - dv - 1)v\,. \end{aligned}$$



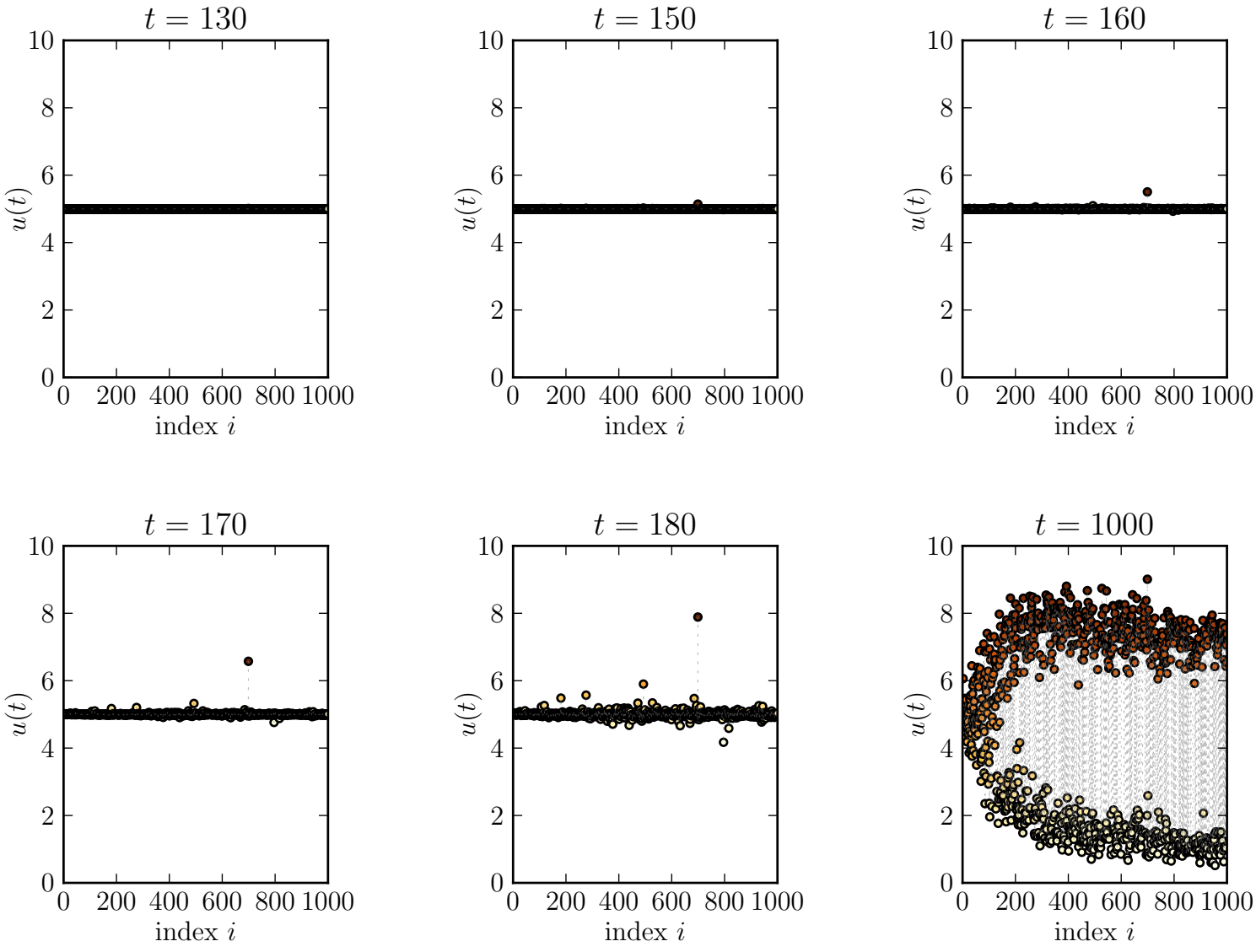
Limit cycle
(a=35, b=16, c=9, c=0.3)



Stable focus $(\bar{u}, \bar{v}) = (5, 10)$
(a=35, b=16, c=9, c=0.4)

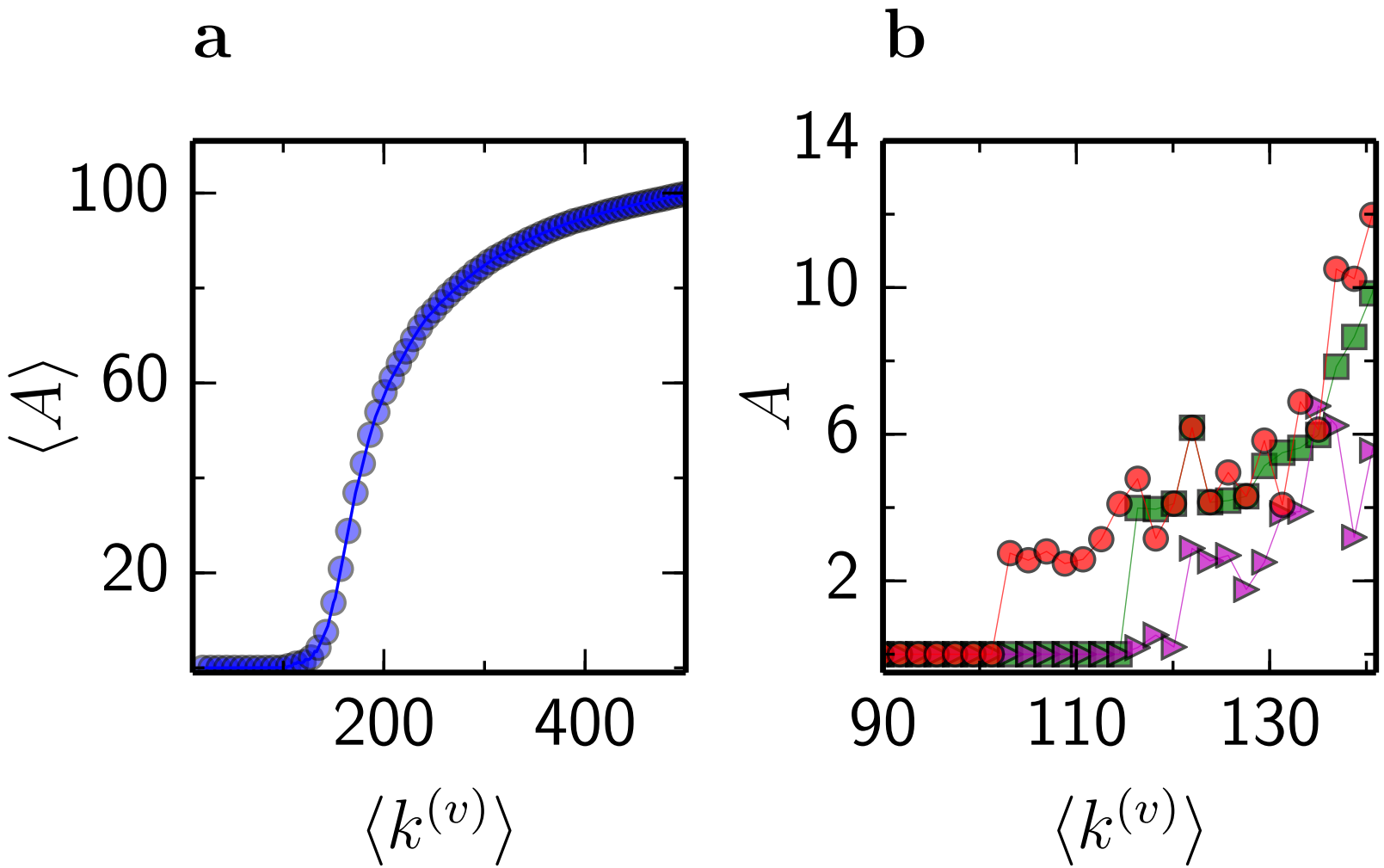
Mimura, Murray J. Theor. Biol. 75 (1978)

Development of the Turing pattern



Kouvaris, Hata, Díaz-Guilera, Sci Rep 5, 10840 (2015)

Amplitude of the Turing pattern



Kouvaris, Hata, Díaz-Guilera, Sci Rep 5, 10840 (2015)

Linear stability analysis

The multiplex organized reaction-diffusion systems reads,

$$\begin{aligned} \dot{u}_i &= f(u_i, v_i) + \sigma^{(u)} \sum_{j=1}^{N} L_{ij}^{(u)} u_j \,, \\ \dot{v}_i &= g(u_i, v_i) + \sigma^{(v)} \sum_{j=1}^{N} L_{ij}^{(v)} v_j \,. \end{aligned}$$

We linearise about the uniform steady state by introducing small perturbations as $(u_i, v_i) = (\bar{u}, \bar{v}) + (\delta u_i, \delta v_i)$

$$\begin{aligned} \dot{\delta u_i} &= f_u \delta u_i + f_v \delta v_i + \sigma^{(u)} \sum_{j=1}^{N} L_{ij}^{(u)} \delta u_j \,, \\ \dot{\delta v_i} &= g_u \delta u_i + g_v \delta v_i + \sigma^{(v)} \sum_{j=1}^{N} L_{ij}^{(v)} \delta v_j \,. \end{aligned}$$

where $f_u = \partial f/\partial u|_{(u,v)=(\bar{u},\bar{v})}$, $f_v = \partial f/\partial v|_{(u,v)=(\bar{u},\bar{v})}$... are partial derivatives at the uniform steady state.

Linear stability analysis

Alternatively, linearized system can be written as,

$$\dot{\mathbf{w}} = (\mathcal{J} + \mathcal{L})\mathbf{w}$$

where:

$$\mathcal{J} = \begin{pmatrix} f_u I & f_v I \\ g_u I & g_v I \end{pmatrix}$$

$$\mathcal{L} = \begin{pmatrix} \sigma^{(u)} L^{(u)} & 0 \\ 0 & \sigma^{(v)} L^{(v)} \end{pmatrix}$$

$\mathbf{w} = (\delta u_1, \cdots, \delta u_N, \delta v_1, \cdots, \delta v_N)^T$ is the perturbation vector; I is the $N \times N$ identity matrix.

Perturbation vector $\mathbf{w}$ should be expanded over the set of eigenvectors of the matrix $\mathcal{Q} = \mathcal{J} + \mathcal{L}$. It is however difficult to calculate them for different network topologies, i.e. different Laplacian matrices $L^{(u)}$ and $L^{(v)}$.

Here we propose an approximation technique to analyze the linear stability.

Linear stability analysis

Matrix $\mathcal{L}$ is splitted as $\mathcal{L} = \mathcal{Q}_0 - \mathcal{D}$, where

$$\mathcal{Q}_0 = \begin{pmatrix} \sigma^{(u)} A^{(u)} & 0 \\ 0 & \sigma^{(v)} A^{(v)} \end{pmatrix}$$

and

$$\mathcal{D} = \begin{pmatrix} \sigma^{(u)} D^{(u)} & 0 \\ 0 & \sigma^{(v)} D^{(v)} \end{pmatrix}$$

Matrices $A^{(u)}$ and $A^{(v)}$ are the adjacency matrices of layers $G^{(u)}$ and $G^{(v)}$, respectively. Matrices $D^{(u)}$ and $D^{(v)}$ are the corresponding degree matrices

Then, matrix $\mathcal{Q}$ can be rewritten as, $\mathcal{Q} = \mathcal{Q}_0 + \mathcal{Q}_1$, where

$$\mathcal{Q}_1 = \mathcal{J} - \mathcal{D} = \begin{pmatrix} f_u I - \sigma^{(u)} D^{(u)} & f_v I \\ g_u I & g_v I - \sigma^{(v)} D^{(v)} \end{pmatrix}$$

Examining matrices $\mathcal{Q}_0$ and $\mathcal{Q}_1$, it can be found that
the first has elements with values of order $\mathcal{O}(\sigma^{(u)})$ or $\mathcal{O}(\sigma^{(v)})$, while
the second has elements with values of order $\mathcal{O}(\sigma^{(u)} k^{(u)})$ or $\mathcal{O}(\sigma^{(v)} k^{(v)})$.

Linear stability analysis: approximation

If both layers are dense enough, i.e. $\langle k^{(u)} \rangle \gg 1$ and $\langle k^{(v)} \rangle \gg 1$, then we can clearly see that the elements of matrix $\mathcal{Q}_1$ take larger enough values than those of matrix $\mathcal{Q}_0$, so that, $\mathcal{Q}_0$ could be neglected yielding to the approximate linearized equation

$$d\mathbf{w}/dt = \mathcal{Q}_1 \mathbf{w}.$$

Then, the characteristic equation for the eigenvalues λ is given by

$$\det \begin{pmatrix} f_u - \sigma^{(u)} k^{(u)} - \lambda & f_v \\ g_u & g_v - \sigma^{(v)} k^{(v)} - \lambda \end{pmatrix} = 0\,,$$

and is the same for each pair of nodes $i^{(v)}$, $i^{(u)}$.

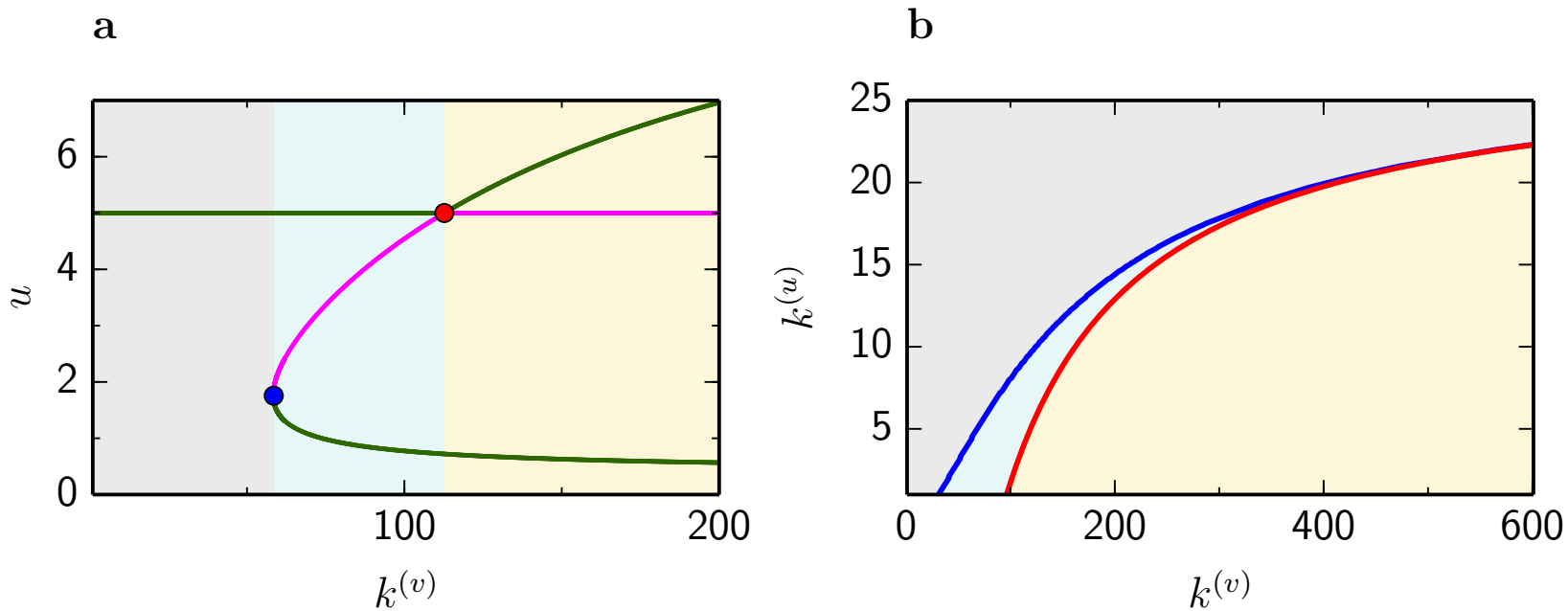
In this approximation, $\mathcal{Q}_0$, which is associated with the precise architecture of the layers, is neglected and, each node is characterized only by its degree.

This is similar to the heterogeneous mean-field method.

Condition for Turing instability

Multiplex topology-driven instability occurs when the degrees of the nodes $i^{(u)}$ and $i^{(v)}$ in the layers $G^{(u)}$ and $G^{(v)}$, respectively, satisfy the condition

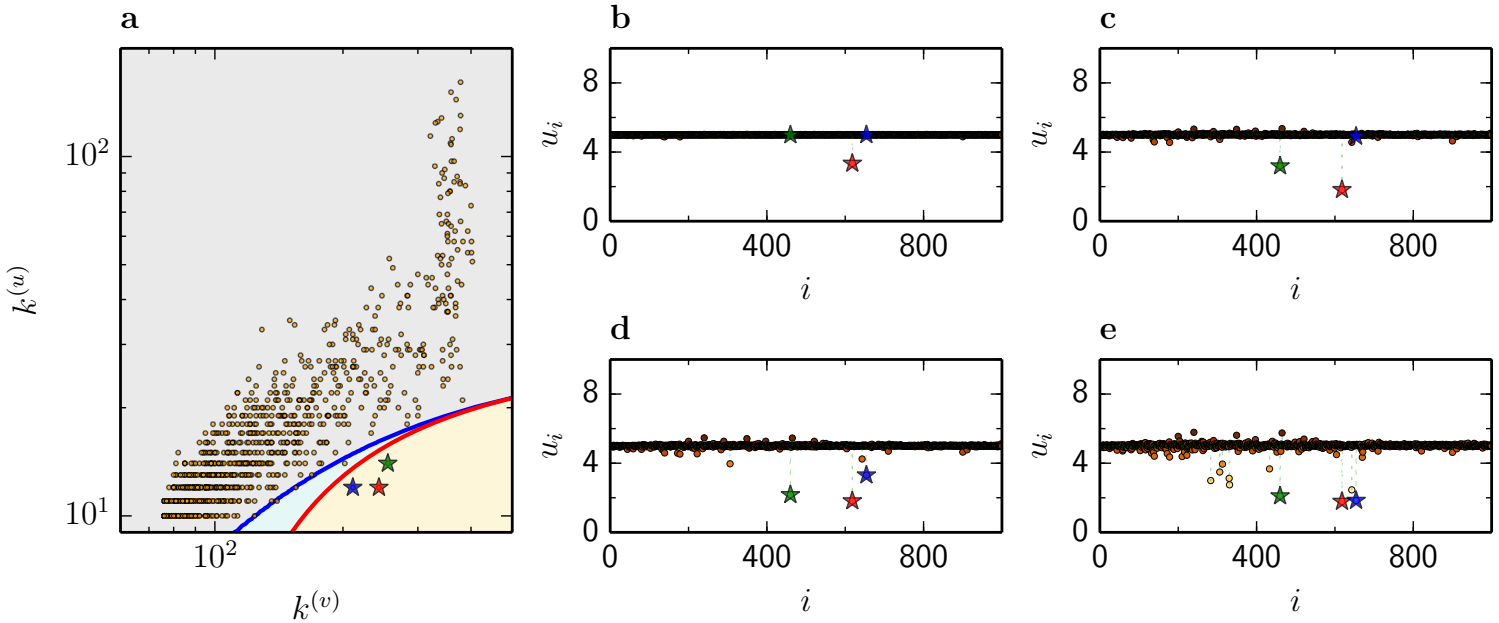
$$k^{(u)} = \frac{f_u g_v - f_v g_u - f_u \sigma^{(v)} k^{(v)}}{g_v \sigma^{(u)} - \sigma^{(u)} \sigma^{(v)} k^{(v)}},$$



Kouvaris, Hata, Díaz-Guilera, Sci Rep 5, 10840 (2015)

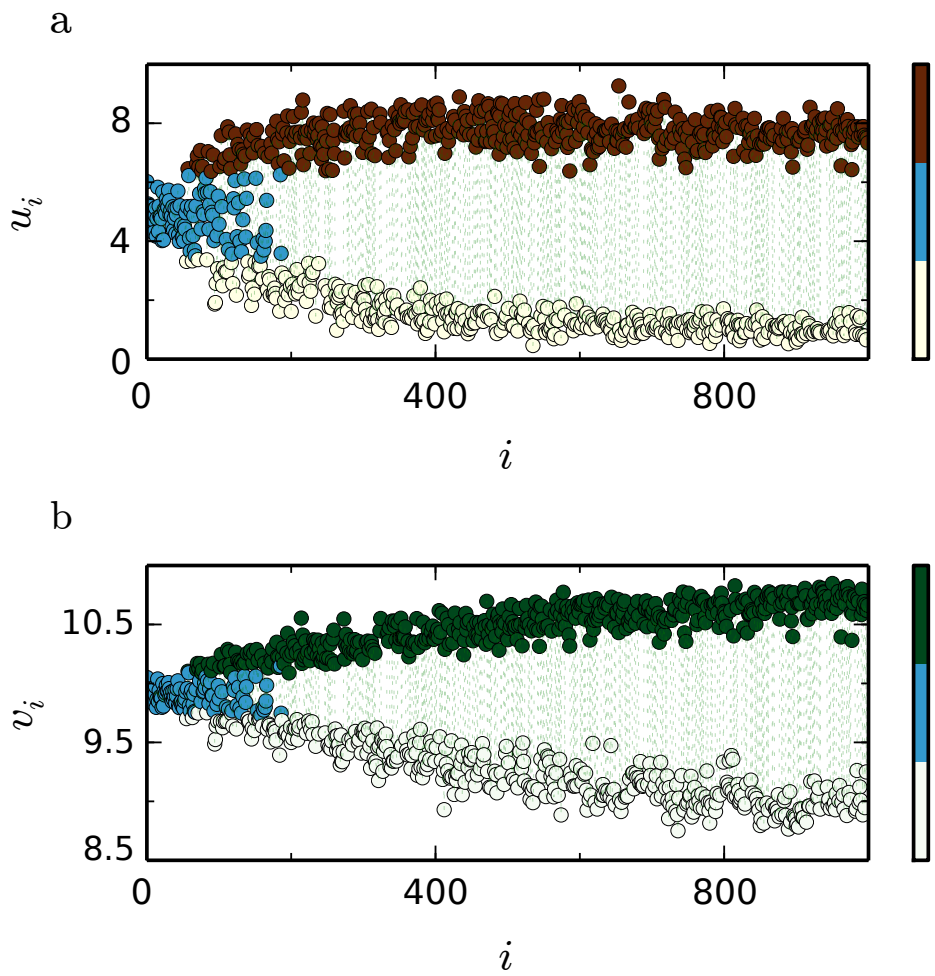
Formation of Turing patterns

$$k^{(u)} = \frac{f_u g_v - f_v g_u - f_u \sigma^{(v)} k^{(v)}}{g_v \sigma^{(u)} - \sigma^{(u)} \sigma^{(v)} k^{(v)}},$$



Kouvaris, Hata, Díaz-Guilera, Sci Rep 5, 10840 (2015)

Actual Turing patterns

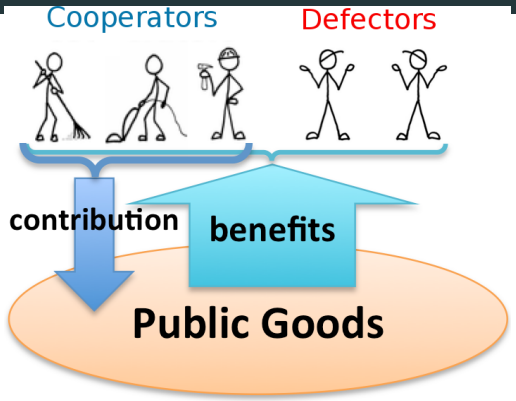


Kouvaris, Hata, Díaz-Guilera, Sci Rep 5, 10840 (2015)

Games on networks

Chimera states are studies in rings of non-locally coupled strategies in public goods games.

Public Goods Games



In the public good game every cooperator yields a benefit r, while defectors produce no benefit getting their share of the public good

The average payoffs read

$$P_{\mathrm{x}} = r\frac{x}{1-z}\left(1-\frac{1-z^n}{n(1-z)}\right)+\frac{r}{n}\frac{1-z^n}{1-z}-1-d\left(\frac{1-z^n}{1-z}-1\right)$$

and

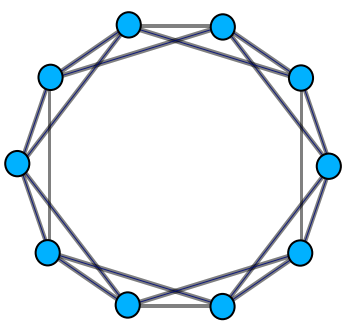
$$P_{\mathrm{y}} = P_{\mathrm{x}} + 1 - \frac{r}{n}\frac{1-z^n}{1-z}$$

and

$$P_{\mathrm{z}} = 0$$

Parameter n is the group size, d is the damage that jokers inflicts on the public good.

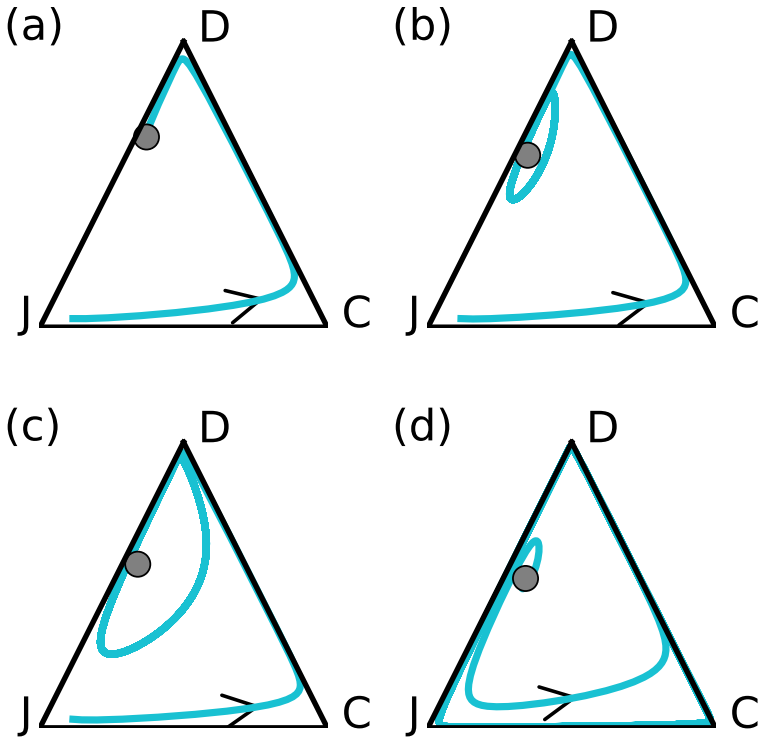
Let us consider the replicator-mutator equations where the stategies of cooperators and defectors are coupled according to a ring architecture,

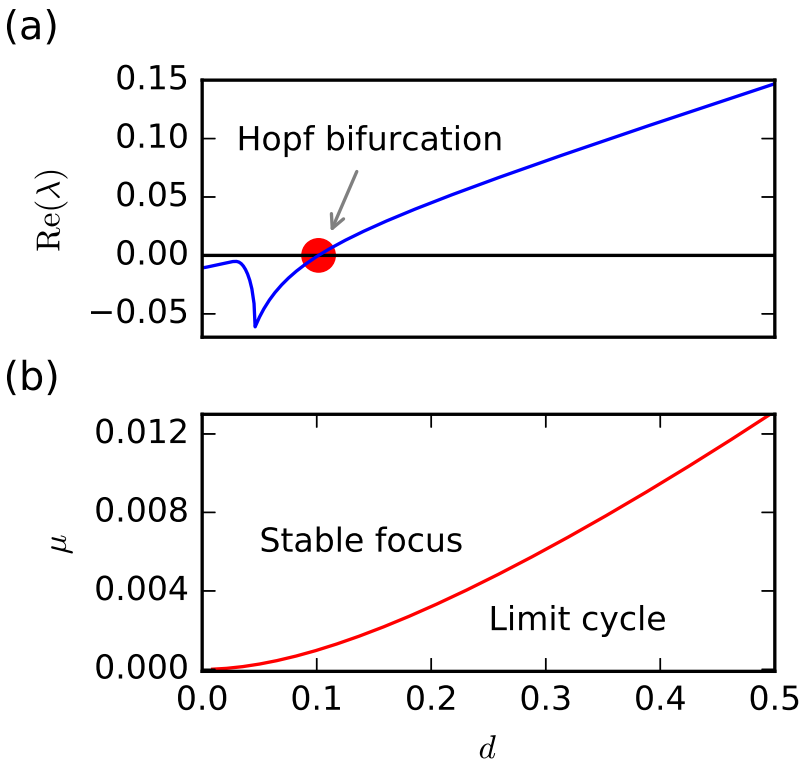


$$\begin{aligned}
\dot{x}_i &= x_i(P_{\mathrm{x},i}-\bar{P}) + \mu(1-3x_i) + \frac{\sigma}{2R}\sum_{j=i-R}^{j=i+R}(x_j-x_i)\,,\\
\dot{y}_i &= y_i(P_{\mathrm{y},i}-\bar{P}) + \mu(1-3y_i) + \frac{\sigma}{2R}\sum_{j=i-R}^{j=i+R}(y_j-y_i)\,,\\
z &= 1-x-y
\end{aligned}$$

where x, y and z are the fractions of cooperators, defectors and jokers, respectively.

Public Goods Games

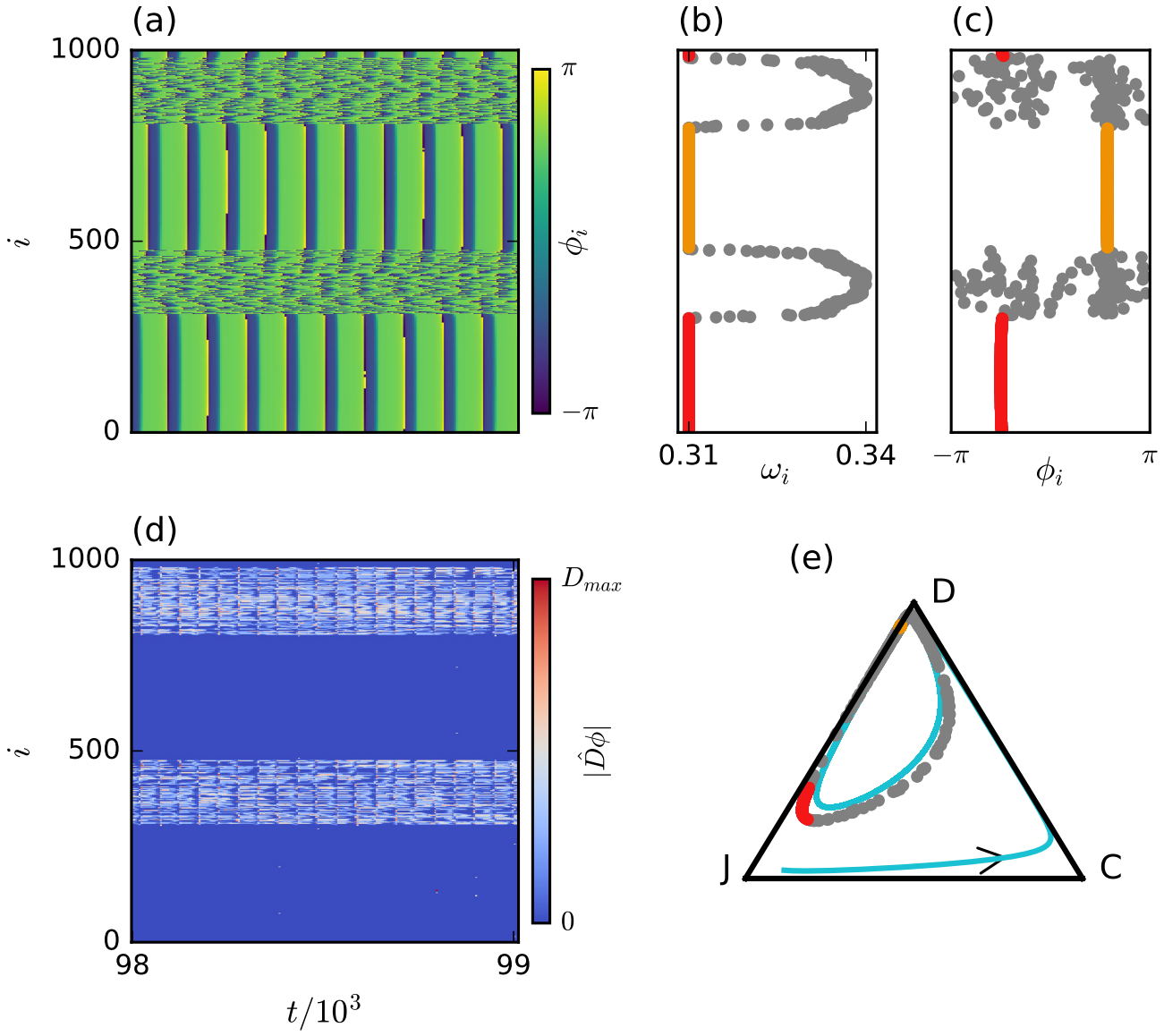




The presence of jokers induces periodically a burst of cooperators

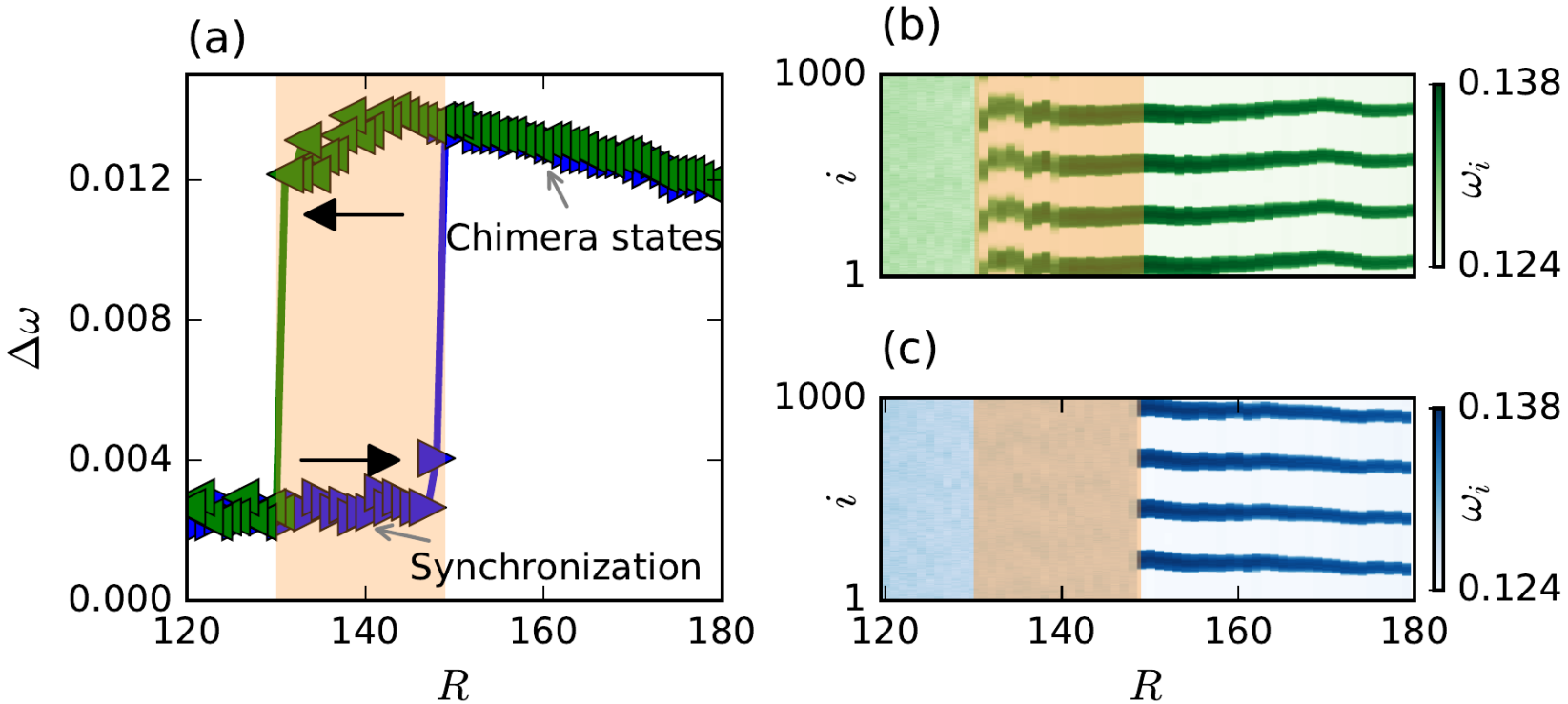
Kouvaris, Requejo, Hizanidis, Díaz-Guilera, CHAOS (2016)

Chimera states in networked PG games



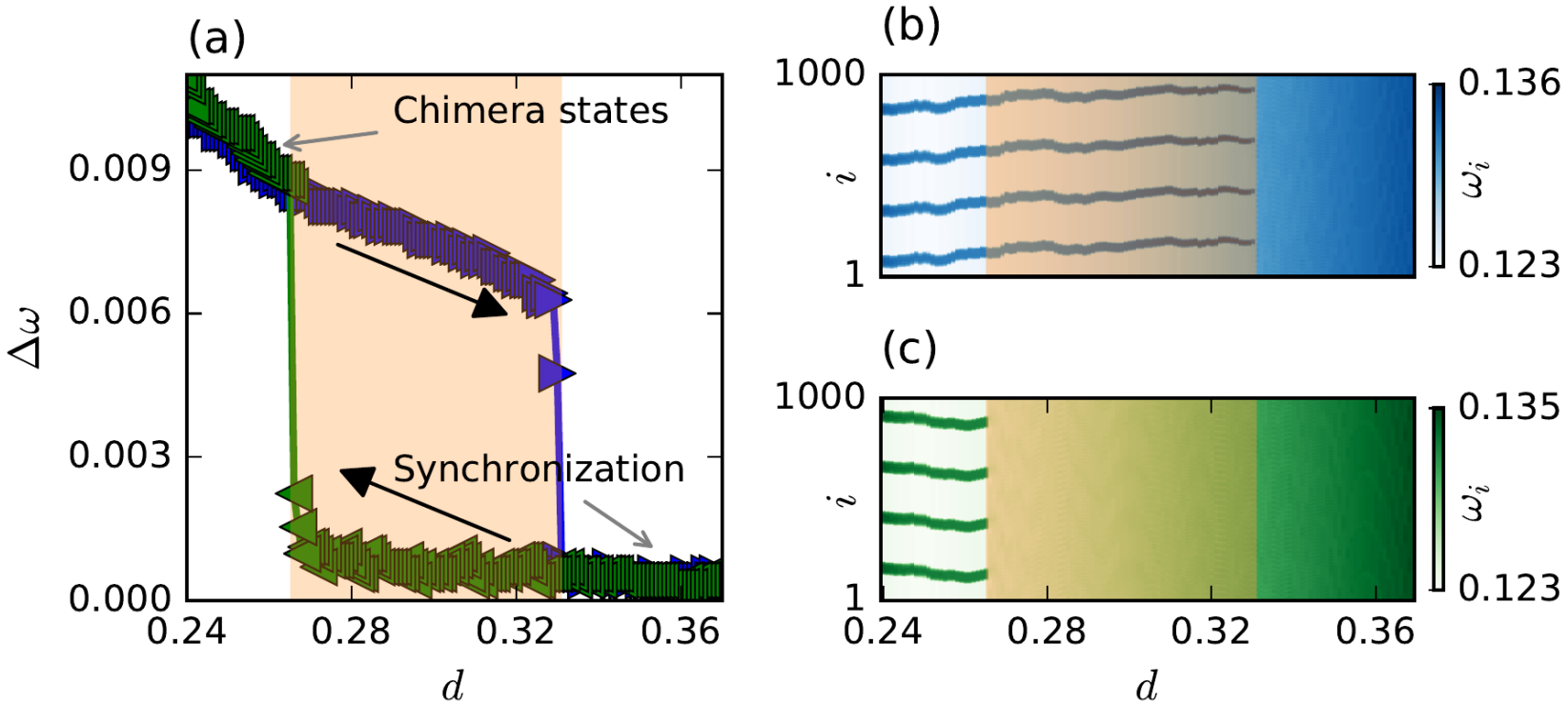
Kouvaris, Requejo, Hizanidis, Díaz-Guilera, CHAOS (2016)

Chimera states in networked PG games



Kouvaris, Requejo, Hizanidis, Díaz-Guilera, CHAOS (2016)

Chimera states in networked PG games



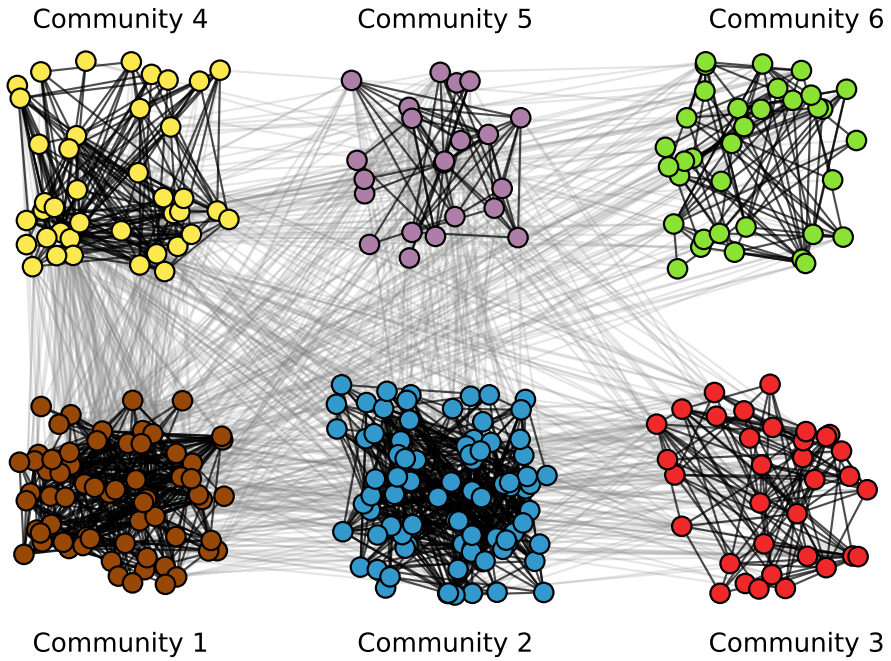
Kouvaris, Requejo, Hizanidis, Díaz-Guilera, CHAOS (2016)

Chimera states in modular neural networks

Chimera states in modular neural networks

C-elegans has two different type of links, chemical (synapses) and electrical (gap junctions). This allows for a multiplex interpretation of its brain, where chimera-like states have been, numerically, found.

The *C-elegans* brain network



Hizanidis, Kouvaris, Zamora-López, Díaz-Guilera, Antonopoulos. Sci Rep (2016)

The Hindmarsh-Rose model:

$$
\begin{aligned}
\dot{p}_i &= q_i - \alpha p_i^3 + b p_i^2 - n_i + I_{\text{ext}} \\
&+ g_l \sum_{j=1}^{N} G_{ij} H(p_j) \\
&- g_n (p_i - V_{\text{syn}}) \sum_{j=1}^{N} C_{ij} S(p_j) \ , \\
\dot{q}_i &= c - d p_i^2 - q_i \ , \\
\dot{n}_i &= r[a(p_i - p_0) - n_i] \ ,
\end{aligned}
$$

where $H(p) = p$ and $S(p) = \frac{1}{1+\exp[-\lambda(p-\Theta_{\text{syn}})]}$ $|p_i| < 2$, thus, $(p_i - V_{\text{syn}})$ is negative for excitatory coupling.

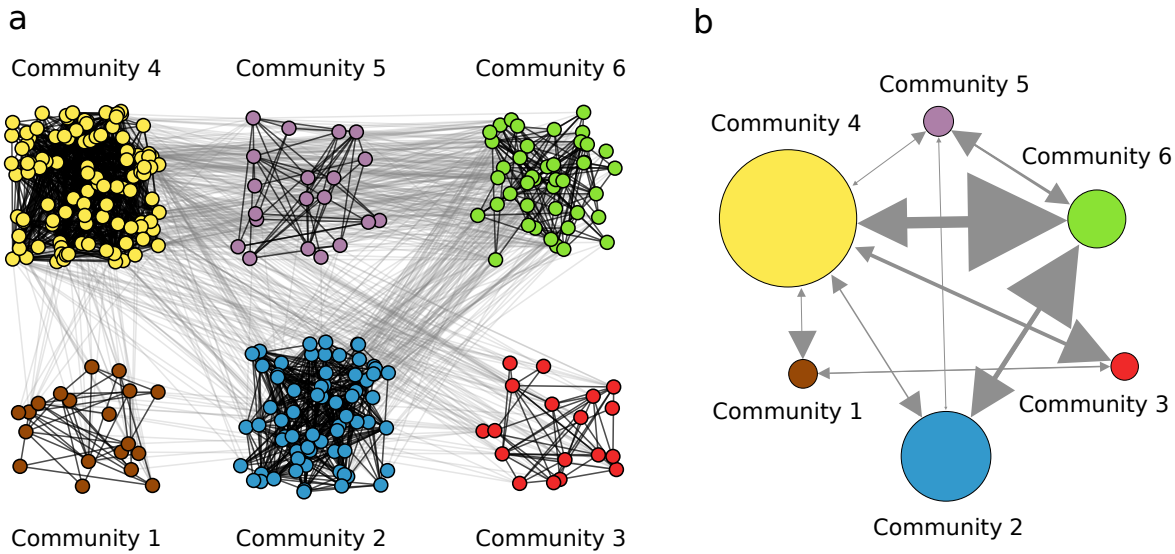
Baptista, Kakmeni, Grebogi, PRE 82 (2010)

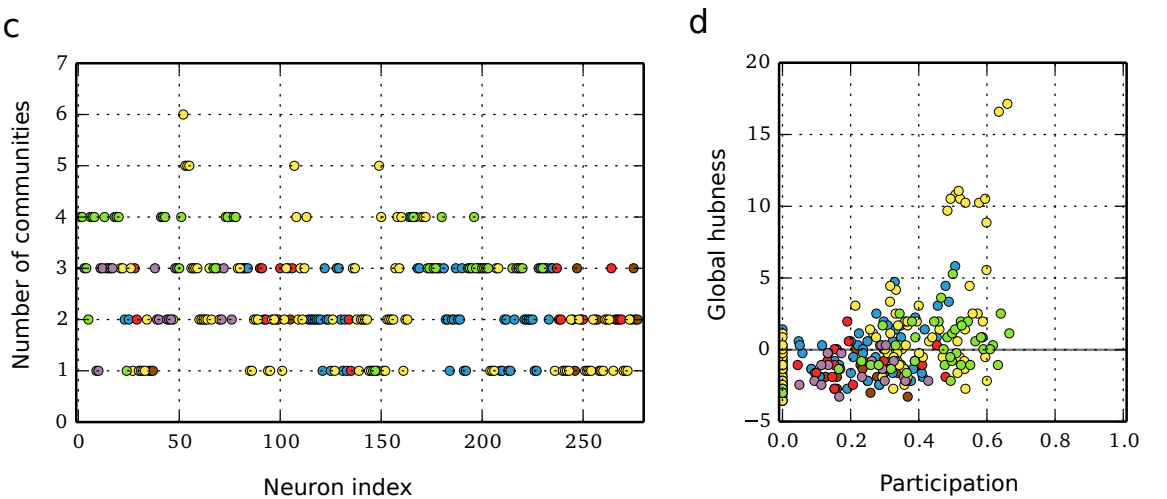
Laplacian G_{ij}: electrical (linear) coupling within each community, $(G_{ij} = A_{ij} - k_i)$

Adjacency C_{ij}: chemical (nonlinear) coupling between the communities.

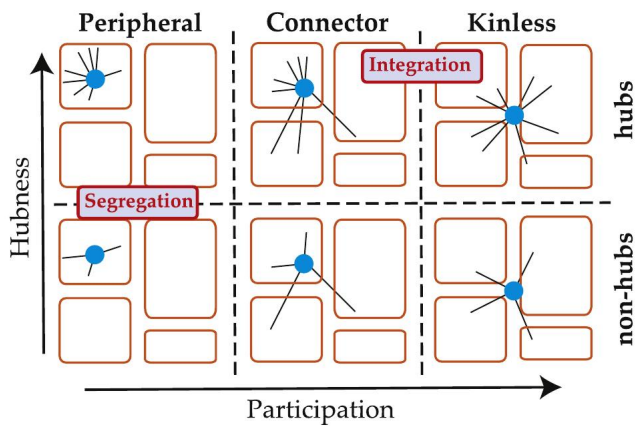
On Monday Myrto Villia will present results using the real chemical and electrical networks

Characterize the modular structure





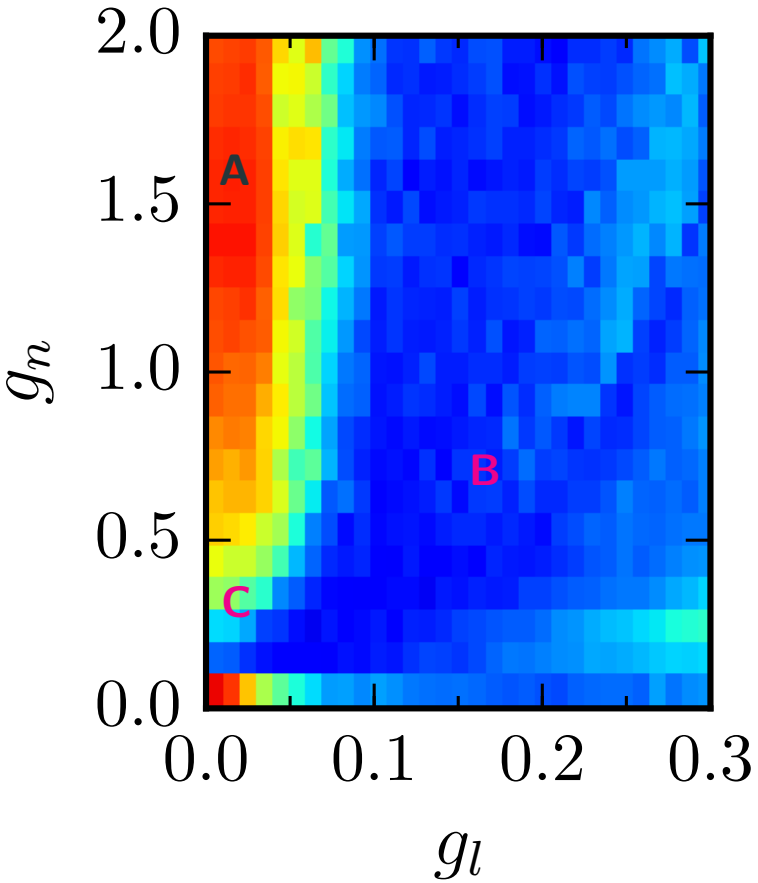
- size of circles is proportional to number of nodes in community
- width of arrow is proportional to number of chemical synapses
- size of arrow head is proportional to number of chemical synapses divided by the mean degree of the target community

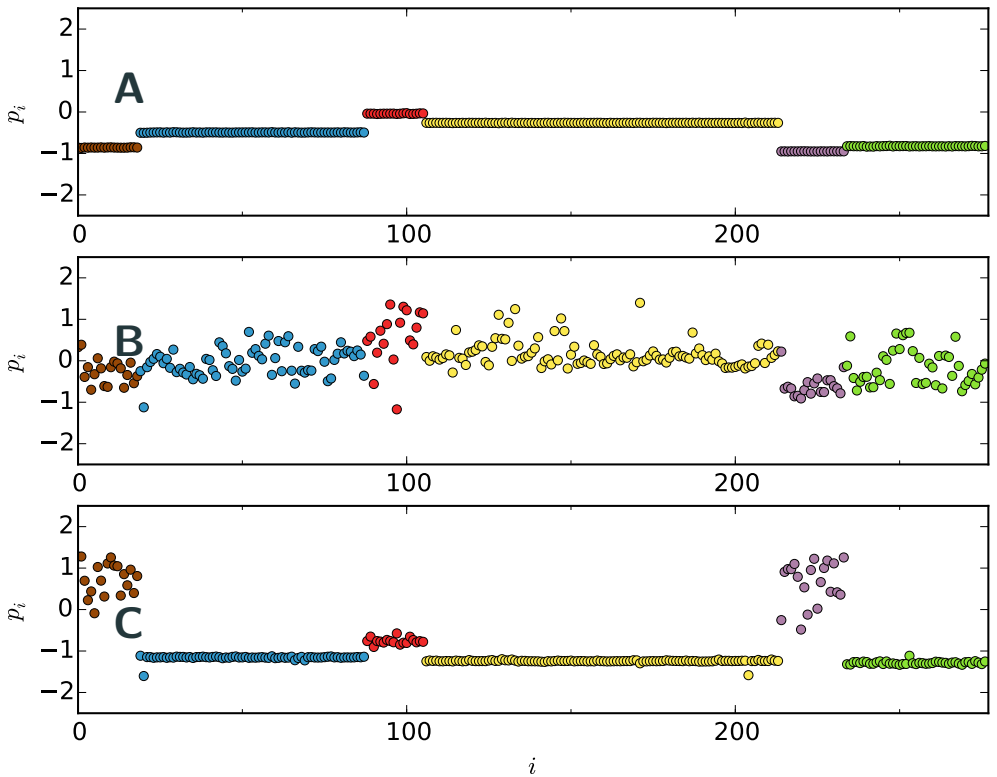


Klimm, Borge-Holthoefer, Wessel, Kurths, Zamora-López, New J. Phys., 2014

Hizanidis, Kouvaris, Zamora-López, Díaz-Guilera, Antonopoulos. Sci Rep (2015)

Chimera-like states in the *c-elegans* brain network





Hizanidis, Kouvaris, Zamora-López, Díaz-Guilera, Antonopoulos. Sci Rep (2015)

Take-home message

Complex networks (simplex, modular, multiplex) support self-organization processes:

1. Traveling fronts
2. Stationary localized patterns
3. Excitation waves
4. Turing patterns
5. Synchronization
6. Chimera sates